Here's a well-thrown disk. But have you ever thrown one badly, so that as it spins, there's a wobble? This wobble rotates with a different frequency than the disk itself spins. In this video, we're going to describe mathematically the motion of all of the points on this badly thrown disk. This video is part of the Linearity Video Series. Many complex systems are modeled or approximated linearly because of the mathematical advantages. Hi, my name is Dan Frey, and I am a professor of Engineering Systems and Mechanical Engineering at MIT. And I use rigid body kinematics myself when designing radio--controlled aircraft. Before watching this video, you should be familiar with eigenvalues and eigenvectors, the standard basis, e1, e2, e3 of R3, and orthogonal matrices. After watching this video, you will be able to: identify rotation matrices; decompose the motion of the badly thrown disk into translational and rotational components; and write the rotational motion of the disk as a product of rotation matrices. Our goal is to describe the motion of the disk. The disk is a rigid body; it doesn't stretch, bend, or deform in any way when it is thrown. In this video, we're not interested in why the disk moves the way it doesâ€"that is, we're not trying to describe torques and forces that govern the motion. We simply want to describe the motion mathematically. This is a job for rigid body kinematics. We're going to describe the motion by decomposing it into translational and rotational components. We'll start with the mathematics of rotation matrices. This will allow us to build up to a description of the wobbly disk. Finally, we'll complete the description of the wobbly disk by adding in the translational component. Let's start with some linear algebra. A rotation is a mapping that takes any vector in R3 to some other vector in R3 via rotation about some axis by some angle. Rotations don't change the length of a vector. So, if you scale a vector and then rotate it, you get the same thing as if you first rotate it, and then scale the vector. Also, if you take two vectors, sum them and then rotate the sum, this is equal to vector you would get if you first rotate both vectors and then add them. These two properties together mean that Rotations act linearly on vectors. And by definition, linear operations can be represented by matrices. But what does a rotation matrix look like? We can learn a lot about a matrix by examining its eigenvalues and eigenvectors. Recall that a vector v is an eigenvector of a matrix if it is sent to a scalar multiple of itself when acted upon by the matrix. That scalar is the eigenvalue. Consider a rotation of 60 degrees about the axis defined by the vector e1+e2. Pause the video here and determine one eigenvalue and eigenvector. By the definition of an eigenvector, the vector e1+e2, which points along the axis of rotation, is an eigenvector with eigenvalue one. This is because this vector is UNCHANGED by the rotation matrix. Suppose you have a rotation matrix such that e1 and e2 are both eigenvectors with eigenvalue 1. What would this mean about the rotation? Pause the video and think about this. The entire xy-plane will be unchanged by this rotation. This is only possible if the matrix is the identity matrix! This is the null rotation... nothing happens! What are the eigenvalues and eigenvectors of a 180-degree rotation about the z-axis? This rotation matrix has one eigenvalue of 1, corresponding to the vector e3, which points along the axis of rotation. But it has more eigenvectors: any vector in the xy-plane is sent to its negative by the rotation, so any vector in the xy-plane is an eigenvector with eigenvalue -1. Now let's consider a rotation by some angle theta (that is not an integer multiple of pi) clockwise about the z-axis. Write a matrix that represents such a rotation. Compute the eigenvalues of this matrix, and use the definition of an eigenvector to explain why this makes sense. You should have found 1 real eigenvalue equal to 1, and two complex conjugate eigenvalues. The real eigenvalue corresponds to the eigenvector e3, which is sent to itself by the rotation, hence the eigenvalue of 1. The fact that the other two eigenvalues are complex means that no other vector is sent to a REAL scalar multiple of itself. This makes sense geometrically because NO other vector points in the same direction it started in after being rotated. Now, how do we describe any general rotation about an arbitrary axis? Well, a matrix is completely defined by how it acts on basis vectors. Since a rotation doesn't change the lengths of vectors or the angles BETWEEN two vectors, a rotated basis will also be a basis for R3! This tells us that any rotation matrix can be described as an orthonormal matrix. The columns are the vectors each standard basis vector is sent to. Is it true that ALL orthonormal matrices rotate vectors? Pause the video. Nope, here's an orthonormal matrix that's not a rotation; it's a reflection. The rule is that only an orthonormal matrix whose determinant is positive 1 is a rotation. But let's get back to thinking about rigid body KINEMATICS. Remember, we want to describe the motion of the disk. We've talked about rotation matrices, but we've left out a very important component: time! How will we describe time dependent rotation? That's right, time dependent matrices. Let's start by modeling a simple motion: the rotation of a disk as it spins clockwise about the positive z-axis. We know how to write a matrix that describes rotation by an angle theta about the z-axis. How would you make this rotation time dependent? Pause the video and discuss. The obvious choice here is to simply make theta a function of time! But how does it depend on time? To write an explicit function, we need to know the rate, omega, at which the disk is rotating. Assume the disk spins with constant angular velocity. We can easily calculate omega by counting the revolutions per second. And there's our matrix for a spinningâ€"but not wobblingâ€"disk. Now let's try a slightly more difficult example. Let's describe the motion of this wobbly, spinning disk as it rotates on this stick. The disk is itself rotating clockwise about its center of mass when viewed from the positive z-axis. As before, we can find the rotation rate, omega-D, of a marked point by counting the revolutions per second. Assume omega_D is constant. Now, observe the slight tilt of the disk off of horizontal. This tilt is created by a rotation about a tilt axis. The tilt axis is the vector in the xy-plane about which the disk is rotated by some small angle theta, creating the tilt. The wobble is created because the tilt axis is rotating clockwise about the positive z-axis. We can visualize this by observing that the normal vector to the disk rotates in a cone shape about the z-axis. By tracking the normal vector's revolutions per second, we can find the rotation rate of the wobble, omega-W, of the normal vector. This is also the rotation rate for tilt axis. We assume omega_W is constant. Notice that omega-D and omega-W are different rotation rates. For simplicity, we assume that the marked point begins along the x-axis; and the initial tilt axis aligns with the x-axis, with the tilt angle theta. Let's start by creating a sequence of rotations that rotates the marked point to the angle omega_D times t and the tilt axis to the angle omega_W times t for any time t. To describe this motion, we are going to decompose the behavior into a sequence of rotations about e1, e2, and e3, which have the benefit of being easy to describe mathematically. I want to start by eliminating the tilt of the disk, so we can imagine it spinning parallel to the ground. What is the matrix that undoes the tilt of theta degrees about the x-axis? Pause and write down a matrix. We rotate by an angle negative theta about the positive x-axis, which is represented by this matrix. Now, I rotate the marked point clockwise about the z-axis by the angle (omega_D minus omega_W) times t. This matrix describes the angle difference traveled by the marked point relative to the position of the tilt axis. Now, we need to make sure that we tilt the disk again so that we can describe the wobble. Since we assume the tilt axis begins along the x-axis, we rotate the disk back to the initial tilted position by theta degrees. This counterclockwise, time-independent rotation about the x-axis is represented by this matrix. Finally we must describe the wobble, created by the rotation of the tilt axis. The tilt axis is rotating clockwise about the positive z-axis with rotation rate omega-W, so at time t, it has rotated by omega_W t degrees, which is what this matrix does. How will we combine these matrices to describe the motion of the marked point? Pause and discuss. We multiply the matrices together. The order we apply each matrix matters. We must perform the rotations in the same order we decomposed the motion, because matrices do not multiply commutatively. Let's understand geometrically why this worked. The angle of the marker is changed in two steps of this process, first a rotation by angle omega-D minus omega-W times t, and then by an angle omega_W times t. In the end, it ends up exactly where it should, at omega-D times t. Only the final matrix affects the tilt axis, rotating it by the angle omega-W times t. Because the disk is a rigid object, by describing the position of the marked point and the tilt axis for all times with matrices, we've actually described the position of every point on the disk. We can find the location of any point at time t by applying this matrix operation to any vector on the initial disk. Now, let's go back to the badly thrown disk. We can apply the rotational transformation directly to our thrown disk. The only changes might be to the rotation rates and the initial position. You can think about how we might change the formula. We'll ignore that. So all that is left to consider is the translation. If you throw a disk and watch it from the side, we can ignore the rotations and focus on the translation of the center point of the disk. For the small time interval that we are interested in describing, the disk moves in a straight, horizontal path. So this vector equation approximates the translation. Because the disk is a rigid object, we get the full description of the motion by simply adding in the translation. To the rotation of the wobbly disk to obtain the following equation of motion for any point on the disk. If you thought this problem was cool, you're not the only one. Richard Feynman studied the kinematics AND the dynamics of the wobbly disk. He was able to show that the rotation rate of the special marked point, omega-D, was exactly twice the rotation rate, omega-W, of the tilt axis. This realization ultimately led to insights into the behavior of electrons. In this video, we saw that rotation matrices are orthogonal matrices with determinant equal to positive 1. The kinematics of rigid bodies involves breaking a problem into translation and rotation. The rotations may be decomposed into several time dependent rotation matrices that are multiplied together. The matrix product added to the translation describes the location at any time of all points on the rigid body. I hope you'll try to describe the motions of various rotating objects that you encounter. Have fun, and good luck!