Divergence
Differential Equations Series

Instructor's Guide

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Developed by the Teaching and Learning Laboratory at MIT
for the Singapore University of Technology and Design

Introduction

When to Use this Video

- In Math 201, in lecture or recitation, after Unit 3: Vector Calculus on the Plane, Lecture 5: Green's Theorem
- In Math 201, in class, during or after Unit 4: Vector Calculus in Space, Lecture 9: The Divergence Theorem
- Prior knowledge: definition of divergence and how to compute divergence

Key Information

Duration: 13:29
Narrators: Prof. Raul Radovitsky and Michelle Nyein, Ph.D. candidate
Materials Needed:
- Paper
- Pencil

Learning Objectives

After watching this video students will be able to determine points at which a vector field is divergent.

Motivation

- Divergence is connected to flux, but because it is easy to compute, this notion is often forgotten within the context of a calculus class.
- Students have a difficult time connecting multivariable calculus concepts to physical examples. This video draws from familiar and unfamiliar examples of physical vector fields, and develops a framework for understand the divergence of these vector fields.
- Physical concepts that are directly connected to divergence, such as incompressibility, sources, and sinks are developed through analysis of the divergence of physical examples.

Student Experience

It is highly recommended that the video is paused when prompted so that students are able to attempt the activities on their own and then check their solutions against the video.

During the video, students will:

- Derive the formula for divergence.
- Predict whether different example vector fields have positive, negative, or zero divergence.

Video Highlights

This table outlines a collection of activities and important ideas from the video.

Time	Feature	Comments
1:37	Chapter 1: Detonation	Prof. Raul Radovitsky describes his work designing materials to protect people from detonation blasts, and explains how the divergence is a fundamental component of a complex modeling problem.
4:02	Chapter 2: Divergence defined	Divergence is defined as a measurement of the net flux per area as the area shrinks to zero.
4:44	Activity: students derive formula for divergence	
6:38	Intuition for divergence in terms of preservation of area or volume introduced.	
7:43	Chapter 3: Examples	Several examples are explored. Students are given the opportunity to predict whether each vector field has positive, negative, or zero divergence.
7:50	Incompressible fluid flow	Simple fluid flow regime from Shapiro Fluid Flow video.
9:07	Student Activity	Students think about what it means for a fluid to be compressible.
9:50	Electric field	
11:13	Rotational field	

Video Summary

This video describes how divergence is a fundamental component of a complex modeling problem involving detonation blasts. The divergence of a vector field is defined physically, and the physical description is connected to the mathematical formula. Students analyze a collection of vector fields to determine whether or not they have positive, negative, or zero divergence by analyzing the change in area or volume of a region of tracer particles.
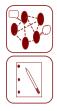
Math 201 Materials

Pre-Video Materials

When appropriate, this guide is accompanied by additional materials to aid in the delivery of some of the following activities and discussions.
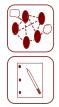
While the divergence theorem is not necessary to understand the main topic of the video, it is needed to understand the continuity equation, which is fundamental to detonations. Because the divergence theorem in some sense defines the divergence operator, we assume it as prior knowledge in 2-dimensions only, and recommend teaching it before watching the video. The problems below both require an ability to apply the divergence theorem to physical situations.

1. The continuity equation is a fundamental partial differential equation that can describe all conservation properties. Break students into small groups. Let the following process to guide students through a derivation the continuity equation in the special case of conservation of mass.

(a) Let ρ be the density or concentration of material in a control volume V. Write an equation for the mass in terms of ρ and V.

(b) If the density depends on time, then within the control volume, the mass also must depend on time. Differentiate this equation to find the rate of change of mass with respect to time as a function of ρ and V.

(c) Let u be the velocity field of the material. Write an integral equation that describes the rate of change in mass as the flux of material through the boundary of the control volume.

(d) Apply the Divergence theorem to rewrite the integral in (c) as the volume integral of the divergence integrand.

(e) Now, set the equation from part (d) equal to the equation found for the change in volume in part (b). Because the control volume was arbitrary, this equation must hold for any control volume. In particular, it holds for infinitesimally small volumes. This can only be true if the two integrands are equal!

This partial differential equation is the continuity equation in the special case of conservation of mass.
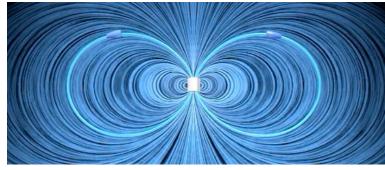
2. The diffusion equation describes diffusion in a wide range of contexts, such as material mixing and heat transfer. Combine the continuity equation describing conservation of mass and Fick's first law (see Gradient video) to obtain the diffusion equation.

Return to examples from Gradient video, and discuss whether or not the examples of Fick's first law were also examples of diffusion. (They are all in fact diffusion examples, because mass is conserved.)

Post-Video Materials

1. Magnetic Field (Appendix A1)

The image above is a 2-dimensional rendering of a magnetic field. Maxwell's equations tell us that magnetic fields are always divergence free. Lead a class discussion about how to consider what would happen to a small region of particles whose velocity field is assumed to be given by the magnetic field vectors.
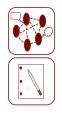
Once students have discussed for a while, ask students to select all statements that best describe what would happen.

(a) A small square region of charged particles would move according to the magnetic field in a way so that the area never changes.

(b) A small cubic region of uncharged particles, whose velocity field is assumed to be given by the magnetic field, would move in such a way that the volume never changes.

(c) A small square region of uncharged particles, whose velocity field is assumed to be given by the magnetic field, would move in such a way that the area never changes.

The key idea in this problem is to make sure that students do not confuse being divergence free in 3-dimensions with being divergence free in 2-dimensions. A magnetic field is divergent free in 3-dimensions, which means that option (b) is the correct item. A student may think that this means that in any projection onto a 2-dimensional plane, the vector field must also be divergence free. But this is not true.

2. Break students into small groups. Have each group find examples of incompressible and compressible materials and fluids. Have them describe how they know they are compressible or incompressible, and how this effects properties of the material or fluid. Have each group share their examples with the full class.

3. Divide students into small groups, and have each student draw vector fields that have positive divergence, negative divergence, and/or zero divergence. Pass vector fields to another member of the group, and have them determine type of divergence. Compare answers, and discuss any ambiguities with the entire class.

4. While deriving and solving the equations for detonation in 3-dimensions is beyond the scope of this class, it may be interesting to derive the equations and some solutions to the equations that would describe 1-dimensional detonation.

It is important to point out to students that this is a fairly common method of problem solving—to first model a problem in 1-dimension—even though this solution will not generalize to higher dimensions, it still helps build intuition.

Additional Resources

Going Further

An understanding of divergence is fundamental to describing physical systems using partial differential equations. By understanding how to use the divergence theorem to obtain such partial differential equations, students will build intuition for the development of arguments involving differential area and volume elements. Such an ability is very important for understanding material properties and statistical thermodynamics.

Two fundamental partial differential equations are the continuity equation and the diffusion equation; these were introduced and derived in the pre and post video activities. However, solutions to these equations were not explored, and is a rich area of study well suited for a numerical methods course.

In the video, the method used to gain an intuition for divergence was adapted from the Reynold's Transport Theorem. The Reynold's Transport Theorem in full generality describes the rate of change of the volume integral of a function or vector field. As a special case, where the boundary of the region moves according to the velocity field, the rate of change of the volume of the region is determined by the divergence of the velocity field. An interesting activity for students in a Fluid Dynamics course would be to explore how the Reynold's Transport Theorem leads to this perspective.

References

The following books give further information about Reynold's Transport Theorem, diffusion, detonations, and divergence in electromagnetism in that order.

- Acheson, D. (1990). *Elementary Fluid Dynamics.* Oxford: Clarendon Press
- Acheson, D. (1997). *From Calculus to Chaos.* Oxford: Oxford University Press.
- Fickett, W. and Davis, W. C. (2000). *Detonation Experiment and Theory.* Mineola, NY: Dover Publications Inc.
- Schey, H. M. (2005). *Div, Grad, Curl and all that: an informal text on vector calculus.* New York, NY: W. W. Norton & Company, Ltd.

The following articles contain information regarding student difficulties with divergence (as well as other vector field operations) as well as some suggested problems and activities around helping students gain a deeper understanding of the operations on vector fields, and how these operations interact.

- Chorlton, F. (1981). Scalar and Vector Functions Considered Fundamentally. *Int. J. of Math. Ed. in Sci. Technol., 12*(1), 175–191.
- Curjel, C.R. (1990). Understanding Vector Fields. *The American Mathematical Monthly, 97*(6), 524–527.
- Markvorsen, S. (2008). The classical version of Stoke's Theorem Revisited. *Int. J. Math. Ed. Sci. Tech., 39*(7), 879–888.

Video references involving divergence on MIT Open CourseWare:

- Auroux, D., *18.02 Multivariable Calculus*, Fall 2007. (Massachusetts Institute of Technology: MIT OpenCouseWare), http://ocw.mit.edu (Accessed March 12, 2012). License: Creative Commons BY-NC-SA

    -Lecture 23: Flux

    -Lectures 28 & 29: The Divergence Theorem

- Open University, (2009). *Div-Grad, Div, and Curl.* (The Open University), http://www.open.edu/openlearn/science-maths-technology/mathematics-and-statistics/grad-div-and-curl?track=d06705765a (Accessed March 13, 2012). License: Creative Commons Attribution-NonCommercial-ShareAlike 2.0 Licence

a) A small square region of charged particles would move in a manner according the the force of the magnetic field such that the area never changes.

b) A small cubic region of uncharged particles, whose velocity is assumed to be given by the magnetic field vectors, would move in such a way that the volume never changes.

c) A small square region of uncharged particles, whose velocity is assumed to be given by the magnetic field vectors, would move in such a way that the are never changes.

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RES.TLL.004 STEM Concept Videos
Fall 2013