Curl

Differential Equations Series

Instructor's Guide

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Developed by the Teaching and Learning Laboratory at MIT
for the Singapore University of Technology and Design

Introduction

When to Use this Video

- In Math 201, in recitation, after Unit 4: Vector Calculus on the Plane, Lecture 5: Green's Theorem
- In Math 201, at home, before Unit 5: Vector Calculus in Space, Lecture 10: Stoke's Theorem
- Prior knowledge: definition of curl and how to compute the curl of a vector field

Key Information

Duration: 12:02
Narrator: Brenden Epps, Ph.D.
Materials Needed:
• Paper
• Pencil

Learning Objectives

After watching this video students will be able to:

- Understand curl as a measurement of the magnitude and direction of maximum circulation per unit area.
- Recognize curl in 2-dimensional fluid flows.
- Describe the relationship between curl and vorticity.
- Connect vorticity to momentum transfer for a collection of familiar physical phenomena.

Motivation

- Students have a difficult time connecting the abstract concept of curl to concrete, observable examples. This video provides several examples rooted in fluid mechanics, along with a framework for thinking about curl.
- This video describes a connection between the curl of a fluid flow and momentum transfer, for example propulsion forces and drag forces.
- The connection between the physical meaning and mathematical description of curl is explored.

Student Experience

It is highly recommended that the video is paused when prompted so that students are able to attempt the activities on their own and then check their solutions against the video.

During the video, students will:

- Recall the physical and mathematical descriptions of divergence, gradient, and curl.
- Derive the formula for a 2-dimensional curl in the xy-plane.
- Predict whether different fluid flow regimes have vorticity.
- Explain why a channel flow has vorticity, given the velocity field.

Video Highlights

This table outlines a collection of activities and important ideas from the video.

Time	Feature	Comments
1:27	Chapter 1: Review of curl	
1:51	Div, Grad, Curl	Activity to recall the mathematical and physical descriptions of these basic calculus operations.
3:22	Activity	Students derive 2-dimensional formula for the circulation per unit area as area approaches zero.
3:42	Animation describing curl in 3-dimensions	
4:22	Activity	Student brainstorm familiar vector fields that have curl.
4:52	Chapter 2: Curl and fluid flows	Vorticity is defined.
5:16	Vorticity meter	The vorticity meter is introduced.
5:44	Rigid body cylindrical rotation	Shapiro video example showing vorticity in the rigid rotation of a cylinder of water, which has been rotating for a very long time.
6:18	Circulating flow in a round tub with a small drain	Shapiro video exploring vorticity in a flow environment with water moving tangentially to a circular tub with a very small drain at the center.
8:08	Channel flow	Shapiro video example showing vorticity in the channel flow.
8:55	Chapter 3: Vorticity and momentum transfer	Drag and propulsion forces are explored.
9:10	Drag	This video of vorticity created in the wake of a cylinder pulled through a fluid. The relationship between the vortices and the drag force is explored.
9:35	Fish swimming	The velocity field of the water surrounding a swimming fish is analyzed to understand how vorticity and momentum transfer are related.
10:55	Robotic fish	This is a florescent dye visualization of the vorticity in the wake of a robot fish created at MIT.

Video Summary

This video leads students through the physical definition of the curl, and its connection to the mathematical description. A variety of classical fluid flow environments are analyzed for the existence of curl. Finally, the connection between curl and momentum transfer in fluid flow are explored in the context of biological locomotion.

Math 201 Materials

Pre-Video Materials

When appropriate, this guide is accompanied by additional materials to aid in the delivery of some of the following activities and discussions.

1. Suppose you have an oddly shaped region in the plane, but you are able to measure the perimeter. Define a vector field on the region that will allow you to determine the area.

This is a classic problem that applies Green's Theorem. A paper that details this problem can be found in the reference section, Davis & Raianu (2007).

2. Rigid Rotation Activity (Appendix A1)

This activity is designed to acquaint students with one of the velocity fields they will encounter in the video. Students will represent the velocity field of a cylinder of water that has been rotating for so long that it moves as a rigid body.

Have students link arms in a line of or 5 students. The student in the center of the line student represents the center of the cylinder. This student acts as the pivot point, and will rotate in place (very slowly). In order to be a rigid flow, students must rotate with the center, maintaining a straight line.

Ask students to describe the velocity of the students rotating in this cylinder. You may want to add more students to the line connected to the central student if students are having a difficult time determining the velocity field of this flow environment.

Post-Video Materials

1. Channel Flow (Appendix A2)

Discuss the channel flow environment as seen in the video.

(a) Where is the curl the largest and where is it the smallest?

(b) Where does the curl change direction?

(c) Is there any place where the curl is zero? How do you know?

2. Circulating Flows (Appendix A3)

Have a discussion about the circulating flow environments seen in the video: the rigidly rotating cylinder of water and the circular drain tub. Ask students to explain why both of these circulating flows must have a region of non-zero curl.

First have students break into small groups and discuss for 10 minutes. If students are having a difficult time understanding what is going on, ask them to find a closed path along which you know the line integral of the velocity field is nonzero. Then have them apply Green's theorem, and what they know about the additivity of integration to take a limit and obtain a region that must have non-zero curl. Then begin a whole class discussion having a variety of groups share their ideas and arguments.

3. Have small groups of students collect examples of video, still images, or demonstrations of physical phenomena where vorticity played a role in an observable effect. Have the groups share their examples with the class.

Some nice examples are fireworks, vortices generated off of mountains, and vortices which produce lift in airplanes. More controversially, vortices are thought to have some effect (although there are many other effects present as well) in "Galloping Gertie", the Tacoma Narrows Bridge collapse. Vortices are also responsible for the audio waves you here by quickly moving a thin rod in air, or a whip cracking.

4. In case you do not have the resources for problem 3, divide students into small groups, and have each student draw vector fields that have curl and do not have curl. Pass them to another member of the group, and have them determine whether or not the field has curl. Compare answers, and discuss any ambiguities with the class.

Additional Resources

Going Further

The curl of a vector field has two major applications that an undergraduate engineering student may encounter: electromagnetism and fluid mechanics. Because students take electromagnetism concurrently with multivariable calculus, the video focuses on examples coming from fluids.

In electromagnetism, the curl is key to understanding the relationship between electric fields and changing magnetic fields, and between magnetic fields and changing electric fields. The differential form of Maxwell's equations describes this relationship using curl. Many physics classes teach these relationships through the integral form instead. However, at the end of Phys 201, it might be interesting to ask students to invoke their new calculus knowledge to write the integral forms of Maxwell's equations in differential form. The resource, Shey (2005), regarding curl and electromagnetism can be found in the references section.

Even though not every student takes a fluid dynamics course, fluid dynamics has applications to a variety of engineering fields. A strong multivariable calculus background is needed to develop a deep understanding of the differential arguments made in such study. Fluid behavior is rarely described at the molecular level, but instead is described using a continuous, differential approximation, the Navier–Stokes equation. While this video does not have the sophisticated language or approach of a fluids course, it may offer a good starting point as it provides a collection of video examples for students starting a fluids course.

A student who is interested in exploring fluid mechanics can be pointed to the Shapiro videos referenced below. In particular, clips from the video on Vorticity were used in this video, and offers an excellent next step to look deeper into this concept.

References

The following educational articles contain descriptions and activities that attempt to describe curl beyond an application of the formula. These resources may provide inspiration in creating your own materials.

- Burch, K. J. & Choi, Y. (2006). The Curl of a Vector Field: Beyond the Formula. *PRIMUS*, *16*(3), 275–287.
- Chorlton, F. (1981). Scalar and Vector Functions Considered Fundamentally. *Int. J. of Math. Ed. in Sci. Technol.*, *12*(1), 175–191.

The following article is a reference for problem 1 in the Pre-Video Materials subsection.

- Davis, P., Raianu, S. (2007). Computing Areas Using Green's Theorem and a Software Planimeter. *Teaching Mathematics and Its Applications: An International Journal of the IMA*, *26*(2), 103–108.

The following article discusses Stoke's theorem in generality, and shows that the Divergence Theorem can be obtained from Stoke's Theorem.

- Markvorsen, S. (2008). The classical version of Stoke's Theorem Revisited. *Int. J. Math. Ed. Sci. Tech.*, *39*(7), 879–888.

Electromagnetism and vector calculus are described in detail in the following text.

- Schey, H. M. (2005). *Div, Grad, Curl and all that: an informal text on vector calculus.* New York, NY: W. W. Norton & Company, Ltd.

These video references discuss curl.

- Auroux, D., *18.02 Multivariable Calculus*, Fall 2007. (Massachusetts Institute of Technology: MIT OpenCouseWare), http://ocw.mit.edu (Accessed March 12, 2010). License: Creative Commons BY-NC-SA
  -Lecture 22: Green's Theorem
  -Lectures 31 & 32: Stoke's Theorem
- Open University, (2009). *Curl-Grad, Div, and Curl.* (The Open University), http://www.open.edu/openlearn/science-maths-technology/mathematics-and-statistics/grad-div-and-curl?track=cf3e7f58b9 (Accessed March 13, 2010). License: Creative Commons Attribution-NonCommercial-ShareAlike 2.0 Licence

The following are references about fluid dynamics. They all discuss vorticity or provide examples of vorticity.

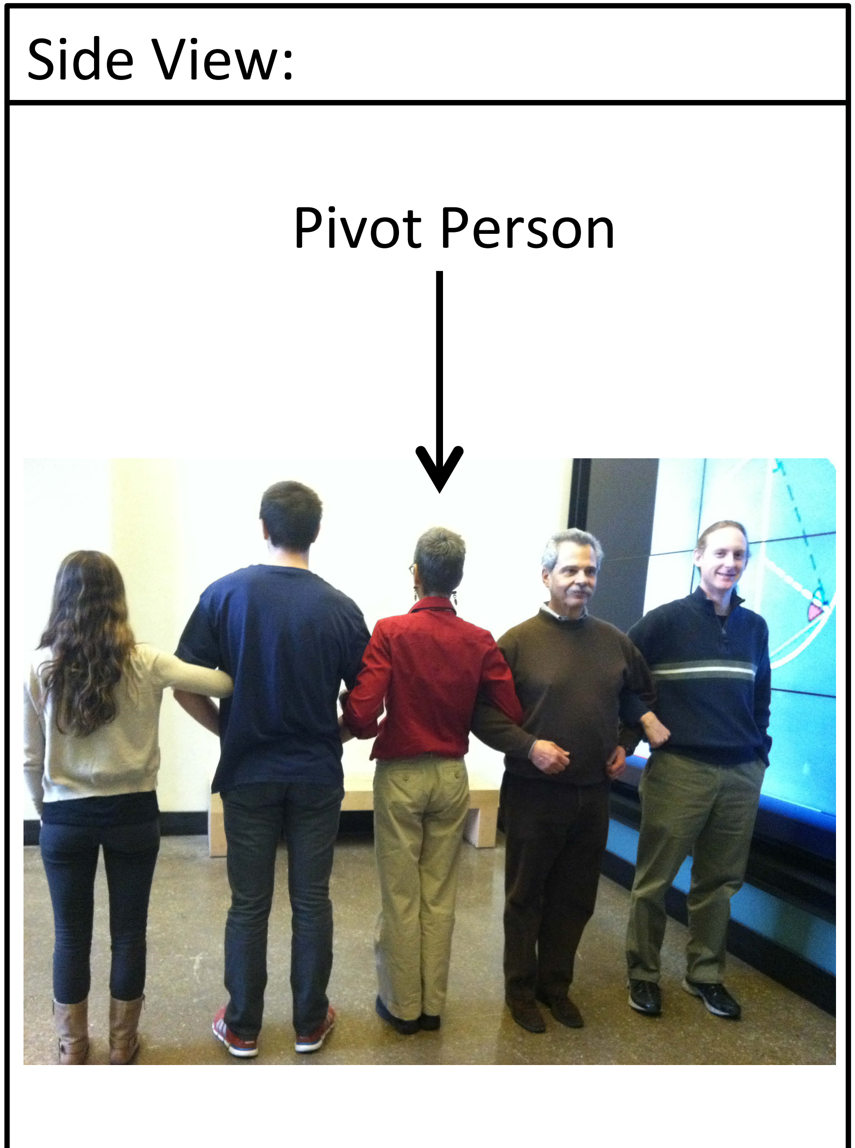
- Acheson, D. J. (1990). *Elementary Fluid Dynamics*. Oxford, England: Clarendon Press.
- Acheson, D. J. (1997). *From Calculus to Chaos: An Introduction to Dynamics.* Oxford, England: Oxford University Press.
- Munson, B. R., Young, D. F., & Okiishi, W. W. (2002). *Fundamentals of Fluid Mechanics, an Interactive E-text.* [CD-ROM], Wiley.
- Olson, R. M. (1980). *Essentials of Engineering Fluid Mechanics*. New York, NY: Harper and Row.
- Van Dyke, M. (1982). *An Album of Fluid Motion*. Stanford, CA: The Parabolic Press.

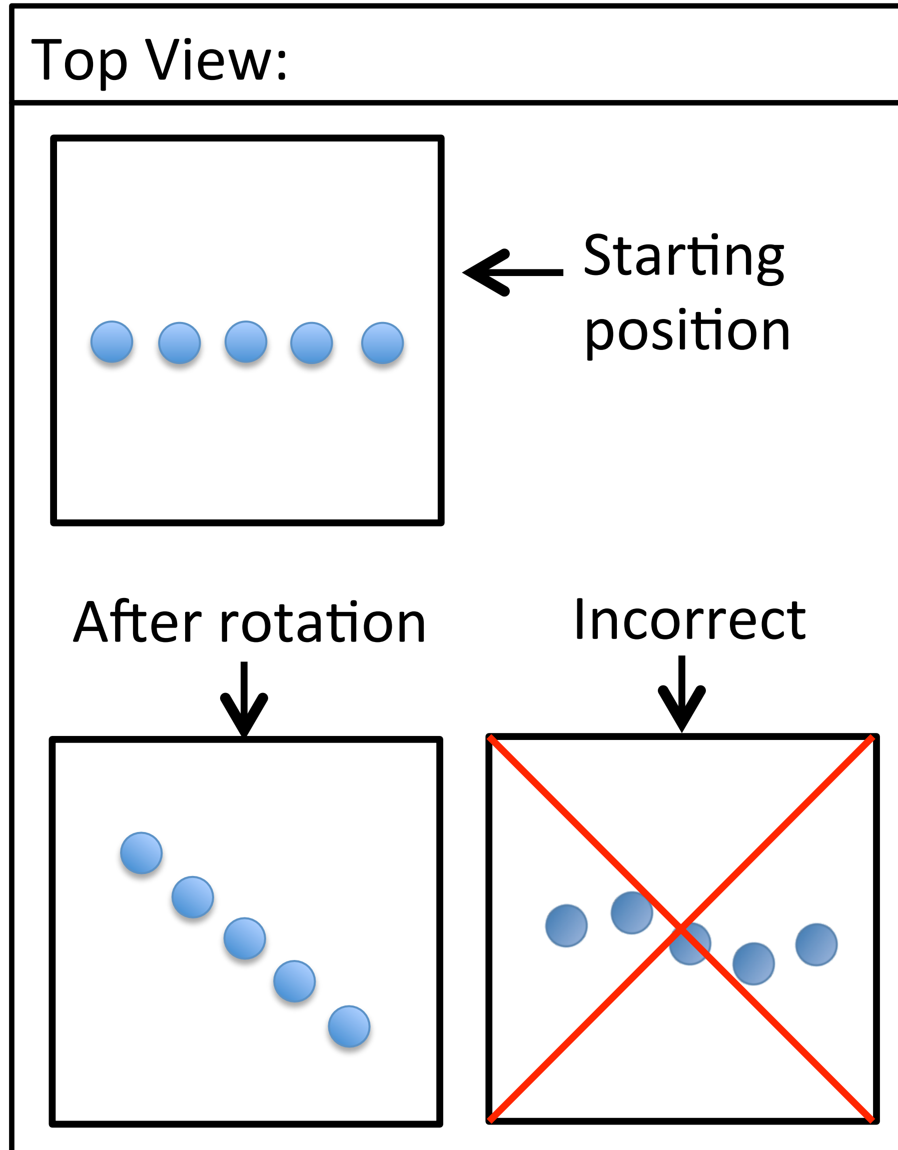
The following videos and associated film notes were created by Prof. Ascher H. Shapiro who founded the National Committee for Fluid Mechanics Films at MIT in collaboration with the Education Development Center. Students or instructors wanting to gain a deeper understanding of vorticity or drag may want to start here.

- Brown, Q. & White, S. (Directors), Smith, K. & Educational Services Incorporated (Producers), (1960). *The Fluid Dynamics of Drag Part I*, [Film]. Available from http://web.mit.edu/hml/ncfmf.html
- Brown, Q. & White, S. (Directors), Smith, K. & Educational Services Incorporated (Producers), (1960). *The Fluid Dynamics of Drag Part II,* [Film]. Available from http://web.mit.edu/hml/ncfmf.html
- Brown, Q. & White, S. (Directors), Smith, K. & Educational Services Incorporated (Producers), (1960). *The Fluid Dynamics of Drag Part III,* [Film]. Available from http://web.mit.edu/hml/ncfmf.html

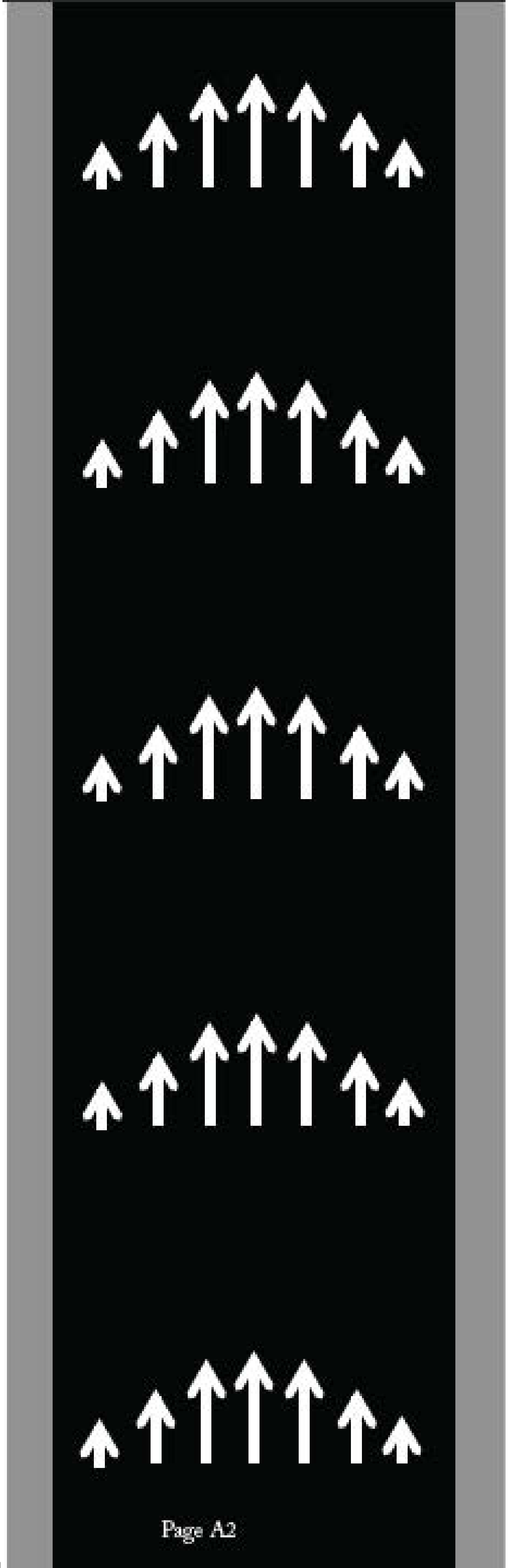
- Churchill, J. & White, S. (Directors) and Smith, K. & Educational Services Incorporated (Producers), (1961). *Vorticity Part I* [Film]. Available from http://web.mit.edu/hml/ncfmf.html
- Churchill, J. & White, S. (Directors) and Smith, K. & Educational Services Incorporated (Producers), (1961). *Vorticity Part II* [Film]. Available from http://web.mit.edu/hml/ncfmf.html

Diagram for rigid rotation activity





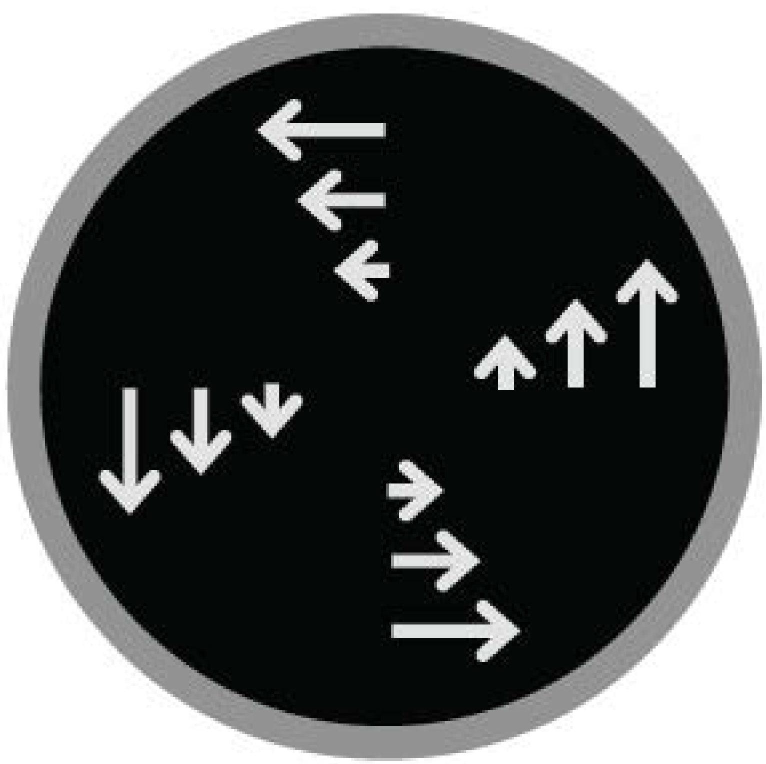
Channel flow velocity field



Channel Flow

Velocity fields of circulating flows

Rigid rotation velocity field



Draining tub velocity field