We will now go through a beautiful example, in which we approach the same question in a number of different ways and see that by reasoning based on the intuitive properties of a Poisson process, we can arrive quickly to the right answer.

The problem is as follows.

We have three lightbulbs, and each light bulb is being lit at time zero, it starts working, and the light bulb lasts for a certain amount of time, which is random.

So this light bulb lasts so long, this one lasts so long, this one lasts that long.

The lifetime of a light bulb, the time until it burns out, will be a random variable, and we make the following assumptions.

The lifetimes of the three light bulbs, which we denote by X, Y, and Z, will be independent random variables, each of which is an exponential random variable with the parameter lambda.

We're interested in the question of calculating the expected time until a light bulb burns out for the first time.

So in this picture, the light bulb that burned out first is the second light bulb, and this quantity is the time until the first burnout.

How do we calculate this quantity?

The time until a first light bulb burns out is the minimum of the times at which each one of them burns out.

So we're interested in the expected value of this quantity.

It's a random variable, which is a function of three random variables.

How do we calculate the expected value of a function of random variables?

We can use the expected value rule.

Let us take the function of interest, which is the minimum of three numbers.

Then we need to multiply by the joint PDF or these three random variables X, Y and Z. Now, because these three random variables are independent, the joint PDF is the product of their individual PDFs, which are all exponential.

And so we obtain this expression for the joint PDF.

And we need to integrate this over all x's, y's, and z's.

So it's going to be an integral that goes for each one of the three variables from 0 to infinity.

And here we have dx, dy, dz.

So this is one approach that can give us the answer if you're able to manipulate and keep track of everything that happens in this three-dimensional integral.

But this is too tedious and this is not a good way to go.

Let us try to think of an alternative way.

Can we figure out the distribution of this random variable?

It's a function of three random variables, so in some sense, it's a derived distribution problem, so we can try to calculate the cumulative distribution function of the minimum of the three.

So for the cumulative, we would be looking at the probability that the minimum of these three random variables is less than or equal to a certain number.

It turns out that the calculations are a little faster if we look at the probability that this is larger than or equal to a certain number.

What is this event?

The minimum is larger than or equal to t, if, and only if, all three of them are larger than or equal to t.

So the probability of this event is the same as the probability that X is larger than or equal to t, Y larger than or equal to t, and Z larger than or equal to t.

But now X, Y, and Z are independent, so we need to multiply the probability that X is larger than or equal to t, which for an exponential random variable is e to the minus lambda t, then the probability of the second event which is again e to the minus lambda t, and, finally, the probability of the third event, which is, again, e to the minus lambda t, which is e to the minus 3 lambda t.

Now, we look at this expression for the probability that the random variable is larger than or equal to t and recognize that this is the expression that we have when we have an exponential random variable with parameter equal to 3 lambda.

If you want to do it formally, subtract this quantity from 1 to find the CDF, take the derivative, and you will find an exponential PDF.

So the conclusion is that this random variable is exponential with parameter 3 lambda.

And since it's an exponential with parameter 3 lambda, then we know what the expected value is.

It is going to be 1 over 3 lambda.

And this is the answer to the question that we were interested in.

Now, the fact that this is an exponential random variable, but with a different parameter, is a pretty clean fact, and so it should have a good explanation.

Let us now try to think about a good explanation for this fact.

Whenever we deal with an exponential random variable, one way of thinking about it is that this exponential random variable is the first time in a Poisson process.

So imagine that there's a Poisson process that runs forever, and X is the first arrival time.

For this light bulb, we can think the same way, and since X and Y are assumed to be independent, we can assume that here we have an independent Poisson process, independent from the first one, it has its own arrival times, and Y is the first arrival time in this Poisson process.

And finally, we have a third independent Poisson process, and the random variable Z is the first arrival time in that Poisson process.

So X, Y and Z are interpreted as first arrivals in three independent Poisson processes.

Now, let us take these three Poisson processes and merge them.

If we merge these three processes, what we obtain is a merged process, which is Poisson with parameter equal to the sum of the rates or parameters of each one of the processes, so it's a Poisson process with parameter 3 lambda.

Now, how can we interpret the random variable of interest, the first burnout time, in terms of the merged process?

So the merged process has an arrival whenever one of those three processes has an arrival.

The first arrival in the merged process will happen the first time that one of these three processes is going to have an arrival.

Therefore, we can interpret the random variable of interest, the first burnout time, as the first arrival time in a merged process.

But now the merged process is Poisson with parameter 3 lambda, therefore, this random variable is going to be an exponential random variable with parameter 3 lambda.

And from this, we also obtain the expected value of that random variable.

The beauty of this last approach for coming up with the answer by reasoning in terms of merged Poisson processes is that we didn't have to do any calculations at all, just use the intuitive understanding of Poisson processes and, especially, the interpretation of an exponential random variable as the first arrival time in a Poisson process.

Let us now try to solve a somewhat harder problem.

Let us try to calculate the expected time until all the light bulbs burn out.

So one light bulb will burn out, then another one will burn out, and, finally, the third one will burn out.

We want to find the expected time until this happens.

Once more, we will be thinking of these burnout times as the first arrival times in Poisson processes.

The total time until the third burnout can be split into different periods.

There's a time until one light bulb burns out.

And the expected value of this period here is going to be 1 over 3 lambda.

What happens at this time?

The second light bulb has burned out, so we can forget about it, take it out of the picture.

We have two lightbulbs.

Let us look at the time it will take until one of these two light bulbs burns out.

So we're interested in this period of time.

Now, the Poisson process starts fresh at this time.

After this time, whatever happens is just an ordinary Poisson process as if it were starting at this time.

So this is going to be an exponential random variable starting from this time.

And this is going to be another exponential random variable.

So the time until the next light bulb burns out in this case is going to be the minimum of two exponential random variables.

We can think again about merging these two Poisson processes to obtain a Poisson process with total rate 2 lambda, and the time until one of these two turns out is going to be the first arrival time in that merged process.

And so the expected time until the first arrival of the merged process is going to be 1 over 2 lambda.

And finally, once this burnout has happened, we can forget about this light bulb.

We're left just with one light bulb, and starting from here, we wait until that light bulb burns out.

Once more, because of the fresh start property of the Poisson process, starting from here until it burns out is going to be a random variable, which is an exponential random variable.

And in this case, an exponential random variable with rate just lambda.

And by adding these three quantities, we get the expected time until all three have burned out.

This is a problem that would have been quite hard to solve in a more analytical way.

We're dealing with a random variable, which is now the maximum of X, Y, and Z. And the distribution of this random variable is not so simple to write down.

So that would not be a very good approach for going about this problem.

But we managed to find the expected value of this random variable without having to write down its distribution, by breaking this random variable into a sum of three particular random variables, each of which had a nice intuitive interpretation.

And that was the key to the solution of this problem.