DESIGN OF FIR DIGITAL FILTERS

Solution 17.1

Rectangular window:

$$W_R(e^{j\omega}) = \sum_{n=-N+1}^{N-1} e^{-j\omega n}$$

$$= \sum_{n=0}^{2N-2} e^{-j\omega(n-N+1)}$$

$$= e^{-j\omega(-N+1)} \sum_{n=0}^{2N-2} e^{-j\omega n}$$

$$= e^{-j\omega(-N+1)} \frac{1-e^{-j\omega(2N-1)}}{1-e^{-j\omega}}$$

$$W_R(e^{j\omega}) = \frac{\sin\left(\left(\frac{2N-1}{2}\right)\omega\right)}{\sin\frac{\omega}{2}}$$

The width of the main lobe is $\frac{4\pi}{2N-1}$ which, for $N \gg 1$ is approximately $\frac{2\pi}{N}$.

Bartlett window:

Define $\tilde{w}_R(n) = 1 \qquad 0 \le n \le N - 1$

$\qquad\qquad = 0 \qquad$ otherwise

Then $\tilde{w}_B(n - N + 1) = \frac{1}{N}\; \tilde{w}_R(n) * \tilde{w}_R(n)$

From Eq. 7.76 of the text (with $N = M + 1$).

$$\tilde{W}_R(e^{j\omega}) = e^{-j\frac{\omega}{2}(N-1)} \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

Therefore

$$W_B(e^{j\omega}) = \frac{1}{N}\left(\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}\right)^2$$

In this case the width of the main lobe is $\frac{4\pi}{N}$ which is twice that for the rectangular window.

Raised cosine window:

$$W_H(e^{j\omega}) = \sum_{-(N-1)}^{N-1} \left[\alpha + \beta\cos\left(\frac{\pi n}{N-1}\right)\right] e^{-j\omega n}$$

$$= \sum_{-(N-1)}^{N-1} \alpha e^{-j\omega n} + \frac{\beta}{2} \sum_{-(N-1)}^{N-1} e^{-jn(\omega+\frac{\pi}{N-1})} + \sum_{-(N-1)}^{N-1} e^{-jn(\omega-\frac{\pi}{N-1})}$$

$$= \alpha \frac{\sin\left(\omega\left(\frac{2N-1}{2}\right)\right)}{\sin\frac{\omega}{2}} + \frac{\beta}{2}\left[\frac{\sin[(\omega+\frac{\pi}{N-1})\frac{2N-1}{2}]}{\sin[\frac{1}{2}(\omega+\frac{\pi}{N-1})]} + \frac{\sin[(\omega-\frac{\pi}{N-1})\frac{2N-1}{2}]}{\sin[\frac{1}{2}(\omega-\frac{\pi}{N-1})]}\right]$$

Assuming that N >> 1, this can be rewritten as

$$W_H(e^{j\omega}) = \alpha \frac{\sin \omega N}{\sin\frac{\omega}{2}} + \frac{\beta}{2}\left[\frac{\sin(N\omega+\pi)}{\sin\frac{1}{2N}(N\omega+\pi)} + \frac{\sin(N\omega-\pi)}{\sin\frac{1}{2N}(N\omega-\pi)}\right]$$

This is the superposition of three terms as sketched in Figure S17.1-1.

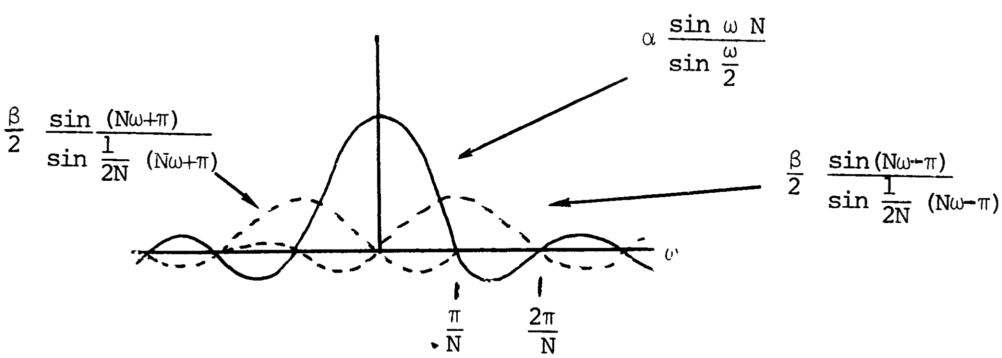

Figure S17.1-1

From this figure we observe that the first values of ω for which the superposition will be zero are $\omega = \pm\frac{2\pi}{N}$. Consequently for this window also, the width of the main lobe is $\frac{4\pi}{N}$.

Solution 17.2

(a) Since $h_1(n)$ and $h_2(n)$ are related by a circular shift, their DFTs are related by

$$H_2(k) = W_8^{4k} H_1(k) = (-1)^k H_1(k) .$$

Thus, their magnitudes are equal.

(b) Since the DFT corresponds to samples of the Fourier transform, the values of $H_1(k)$ are the samples of $H_1(e^{j\omega})$ indicated in Figure S17.2-1

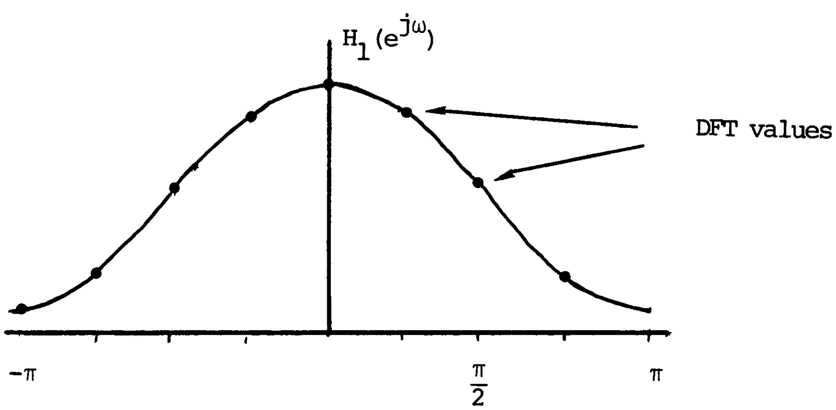

Figure S17.2-1

From (a), $H_2(k) = (-1)^k H_1(k)$. Thus the values $H_2(k)$ are as indicated in Figure S17.2-2. Since these alternate in polarity, the continuous frequency response of which these are samples, must go through zero in between these samples as indicated.

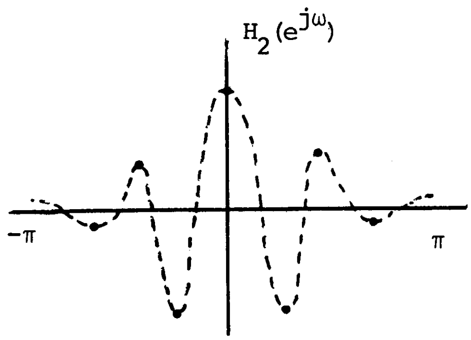

Figure S17.2-2

Thus $h_2(n)$ obviously does not correspond to a good low pass filter, even though the magnitude of its DFT values are identical to those of the low pass filter $h_1(n)$.

Solution 17.3

(a) Since $E(e^{j\omega}) = H_d(e^{j\omega}) - H(e^{j\omega})$, and e(n) is the inverse Fourier transform of $E(e^{j\omega})$,

$$e(n) = h_d(n) - h(n).$$

(b) $$\varepsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |E(e^{j\omega})|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} e(n)e(k)e^{-j\omega n}e^{j\omega k}\right] d\omega \quad .$$

Interchanging the order of integration and summation

$$\varepsilon^2 = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} e(n)e(k)\underbrace{\int_{-\pi}^{\pi} e^{j\omega(k-n)} d\omega}_{\substack{=0 \quad k \neq n \\ =2\pi \quad k = n}}$$

Thus,

$$\varepsilon^2 = \sum_{n=-\infty}^{+\infty} e^2(n) \; .$$

(c) From the results of parts (a) and (b),

$$\varepsilon^2 = \sum_{n=-\infty}^{+\infty} [h_d(n)-h(n)]^2$$

or, since $h(n) = 0$, $n < 0$ and $n \geq N$,

$$\varepsilon^2 = \sum_{n=0}^{N-1} [h_d(n)-h(n)]^2 + \sum_{n=-\infty}^{-1} h_d^2(n) + \sum_{n=N}^{\infty} h_d^2(n) \; .$$

Clearly, the choice of $h(n)$ cannot affect the last summations. The first is non-negative and consequently its minimum value is zero which is achieved for $h(n) = h_d(n)$.

It should be stressed that although a rectangular window minimizes the mean square error, it does not generally result in the best frequency characteristic in terms, for example of passband or stopband ripple.

Resource: Digital Signal Processing
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