Solutions Manual for Electromechanical Dynamics

PROBLEM 3.1

This problem is a simple extension of that considered in Sec. 3.2, having the purpose of emphasizing how the geometric dependence of the electrical force depends intimately on the electrical constraints.

Part a

The system is electrically linear. Hence, $W_m' = \frac{1}{2} L i^2$ and the force f that must be applied to the plunger is

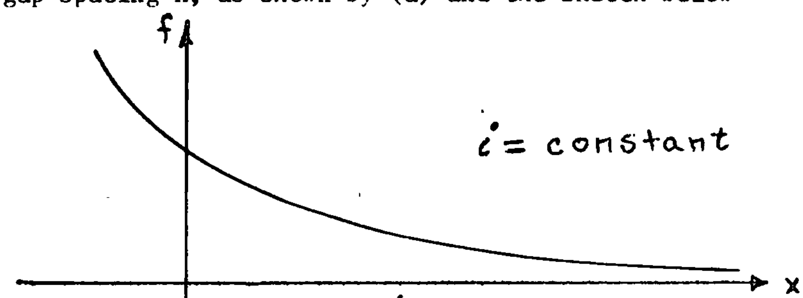
$$f = -f^e = \frac{1}{2a} \frac{L_o i^2}{(1 + \frac{x}{a})^2} \tag{a}$$

The terminal equation can be used to write this force in terms of λ

$$f = -f^e = \lambda^2/2aL_o \tag{b}$$

Part b

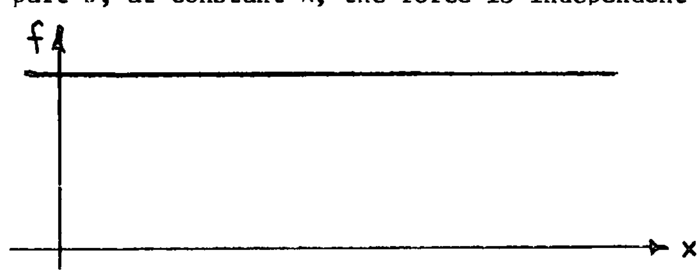
With the current constant, the force decreases rapidly as a function of the plunger gap spacing x, as shown by (a) and the sketch below



With the current constant, the drop in $\int \bar{H}\cdot d\bar{\ell}$ across the gap increases with x, and hence the field in the gap is reduced by increasing x.

Part c

By contrast with part b, at constant λ, the force is independent of x

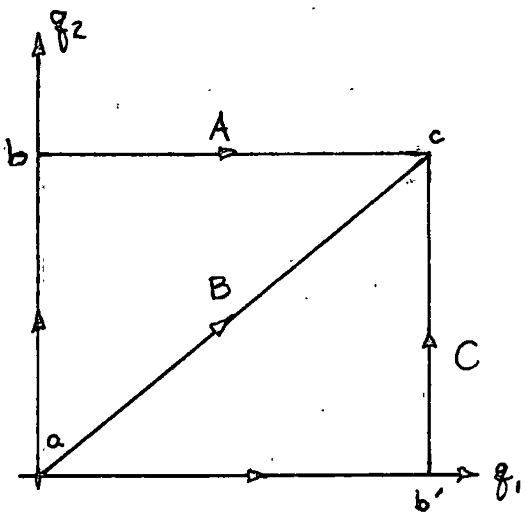


PROBLEM 3.1 (Continued)

With this constraint, the field in the gap must remain constant, independent of the position x.

PROBLEM 3.2

Part a



The terminal relations are

$$v_1 = S_{11}q_1 + S_{12}q_2$$
$$v_2 = S_{21}q_1 + S_{22}q_2 \qquad \text{(a)}$$

Energy input can result only through the electrical terminal pairs, because the mechanical terminal pairs are constrained to constant position. Thus,

$$W_e = \int v_1 dq_1 + v_2 dq_2 \qquad \text{(b)}$$

First carry out this line integral along the contour A: from a→b, $q_1 = 0$, while from b→c, $dq_2 = 0$. Hence,

$$W_e = \int_o^{Q_2} v_2(0,q_2)dq_2 + \int_o^{Q_1} v_1(q_1,Q_2)dq_1 \qquad \text{(c)}$$

and using (a),

$$W_e = \int_o^{Q_2} S_{22}q_2 dq_2 + \int_o^{Q_1}(S_{11}q_1 + S_{12}Q_2)dq_1 \qquad \text{(d)}$$

and for path A,

$$W_e = \frac{1}{2} S_{22}Q_2^2 + S_{12}Q_1Q_2 + S_{11}Q_1^2 \qquad \text{(e)}$$

If instead of path A, we use C, the roles of q_1 and q_2 are simply reversed. Mathematically this means 1→2 and 2→1 in the above. Hence, for path C

$$W_e = \frac{1}{2} S_{11}Q_1^2 + S_{21}Q_2Q_1 + S_{22}Q_2^2 \qquad \text{(f)}$$

To use path B in carrying out the integration of (b), we relate q_2 and q_1

PROBLEM 3.2 (Continued)

$$q_2 = \frac{Q_2}{Q_1} q_1 \tag{g}$$

Then, (a) becomes,

$$v_1 = [S_{11} + \frac{S_{12}Q_2}{Q_1}]q_1; \quad v_2 = [S_{21} + \frac{S_{22}Q_2}{Q_1}]q_1 \tag{h}$$

and, from (b), where dq_2 and Q_2dq_1/Q_1

$$W_e = \int_o^{Q_1} [S_{11} + \frac{S_{12}Q_2}{Q_1}]q_1 dq_1 + \int_o^{Q_1} [S_{21} + \frac{S_{22}Q_2}{Q_1}]q_1 \frac{Q_2}{Q_1} dq_1 \tag{i}$$

or

$$W_e = \frac{1}{2} S_{11}Q_1^2 + \frac{1}{2} S_{12}Q_2Q_1 + \frac{1}{2} S_{21}Q_1Q_2 + \frac{1}{2} S_{22}Q_2^2 \tag{j}$$

Part b

The integrations along paths A, B and C are the same only if $S_{21} = S_{12}$ as can be seen by comparing (e), (f) and (j).

Part c

Conservation of energy requires

$$dW_e(q_1,q_2) = v_1dq_1 + v_2dq_2 = \frac{\partial W_e}{\partial q_1} dq_1 + \frac{\partial W_e}{\partial q_2} dq_2 \tag{k}$$

Since q_1 and q_2 are independent variables

$$v_1 = \frac{\partial W_e}{\partial q_1} ; \quad v_2 = \frac{\partial W_e}{\partial q_2} \tag{ℓ}$$

Taking cross derivatives of these two expressions and combining gives

$$\frac{\partial v_1}{\partial q_2} = \frac{\partial v_2}{\partial q_1} \tag{m}$$

or, from (a), $S_{12} = S_{21}$.

PROBLEM 3.3

The electric field intensity between the plates is

$$E = v/a \tag{a}$$

PROBLEM 3.3 (Continued)

Hence, the surface charge adjacent to the free space region on the upper plate is

$$\sigma_f = \varepsilon_o\, v/a \qquad \text{(b)}$$

while that next to the nonlinear dielectric slab is

$$\sigma_f = \alpha\,\frac{v^3}{a^3} + \varepsilon_o\,\frac{v}{a} \qquad \text{(c)}$$

It follows that the total charge on the upper plate is

$$q = \frac{dx\varepsilon_o v}{a} + d(\ell-x)\left[\frac{\alpha v^3}{a^3} + \frac{\varepsilon_o v}{a}\right] \qquad \text{(d)}$$

The electric co-energy is

$$W_e' = \int q\,dv = \frac{d\ell\varepsilon_o v^2}{2a} + \frac{d(\ell-x)\alpha v^4}{4a^3} \qquad \text{(e)}$$

Then, the force of electrical origin is

$$f^e = \frac{\partial W_e'}{\partial x} = -\frac{d\alpha v^4}{4a^3} \qquad \text{(f)}$$

PROBLEM 3.4

Part a

The magnetic field intensity in the gap must first be related to the excitation current. From Ampere's law,

$$Ni = dH_d + xH_x \qquad \text{(a)}$$

where the fields H_d and H_x are directed counterclockwise around the magnetic circuit when they are positive. These fields are further related because the magnetic flux into the movable member must equal that out of it

$$\mu_o wbH_d = \mu_o waH_x \qquad \text{(b)}$$

From these two expressions

$$H_x = Ni/(\frac{da}{b} + x) \qquad \text{(c)}$$

PROBLEM 3.4 (Continued)

The flux linked by the electrical terminals is $\lambda = N\mu_o awH_x$ which in view of (c) is

$$\lambda = Li; \; L = \frac{N^2\mu_o aw}{(\frac{da}{b} + x)} \tag{d}$$

Part b

The system is electrically linear. Hence, $W_m = \frac{1}{2}\lambda^2/L$ (See Sec. 3.1.2b) and from (d),

$$W_m = \frac{1}{2}\lambda^2 \frac{(\frac{da}{b} + x)}{N^2\mu_o aw} \tag{e}$$

Part c

From conservation of energy $f^e = -\partial W_m/\partial x$, $W_m = W_m(\lambda,x)$. Hence,

$$f^e = -\frac{1}{2}\lambda^2/(N^2\mu_o aw) \tag{f}$$

Part d

In view of (d) the current node equation can be written as (remember that the terminal voltage is $d\lambda/dt$)

$$I(t) = \frac{1}{R}\frac{d\lambda}{dt} + \frac{\lambda(\frac{da}{b} + x)}{N^2\mu_o aw} \tag{g}$$

Part e

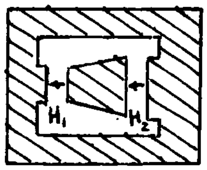
The inertial force due to the mass M must be equal to two other forces, one due to gravity and the other f^e. Hence,

$$M\frac{d^2x}{dt^2} = Mg - \frac{1}{2}\frac{\lambda^2}{N^2\mu_o aw} \tag{h}$$

(g) and (h) are the required equations of motion, where (λ,x) are the dependent variables.

PROBLEM 3.5

Part a



From Ampere's Law

$$H_1(a+x) + H_2(a-x) = N_1i_1 + N_2i_2$$

Because $\oint_S \bar{B}\cdot\bar{n}da = 0$

$$\mu_oH_1A_1 = \mu_oH_2A_2$$

solving for H_1

$$H_1 = \frac{N_1i_1 + N_2i_2}{a(1 + \frac{A_1}{A_2}) + x(1 - \frac{A_1}{A_2})}$$

Now the flux ϕ in each air gap must be the same because

$$\phi = \mu_oH_1A_1 = \mu_oH_2A_2$$

and the flux linkages are determined to be $\lambda_1 = N_1\phi$ and $\lambda_2 = N_2\phi$. Using these ideas

$$\lambda_1 = N_1^2L(x)i_1 + N_1N_2L(x)i_2$$

$$\lambda_2 = N_2N_1L(x)i_1 + N_2^2L(x)i_2$$

where $$L(x) = \frac{\mu_oA_1}{a(1 + \frac{A_1}{A_2}) + x(1 - \frac{A_1}{A_2})}$$

Part b

From part a the system is electrically linear, hence

$$W_m' = L(x)[\frac{1}{2}N_1^2i_1^2 + N_1N_2i_1i_2 + \frac{1}{2}N_2^2i_2^2]$$

where $$L(x) = \frac{\mu_oA_1}{a(1 + \frac{A_1}{A_2}) + x(1 - \frac{A_1}{A_2})}$$

PROBLEM 3.6

Part a

Conservation of energy requires that

$$dW = i d\lambda - f^e dx \tag{a}$$

In addition,

$$dW = \frac{\partial W}{\partial \lambda} d\lambda + \frac{\partial W}{\partial x} dx \tag{b}$$

so that

$$i = \frac{\partial W}{\partial \lambda} \; ; \; f^e = - \frac{\partial W}{\partial x} \tag{c}$$

Now if we take cross-derivatives of these last relations and combine,

$$\frac{\partial i}{\partial x} = - \frac{\partial f^e}{\partial \lambda} \tag{d}$$

This condition of reciprocity between the electrical and mechanical terminal pairs must be satisfied if the system is to be conservative. For the given terminal relations,

$$\frac{\partial i}{\partial x} = - \frac{I_o}{a}\left[\frac{\lambda}{\lambda_o} + \left(\frac{\lambda}{\lambda_o}\right)^3\right]/\left(1 + \frac{x}{a}\right)^2 \tag{e}$$

$$-\frac{\partial f^e}{\partial \lambda} = - \frac{I_o}{a}\left[\frac{\lambda}{\lambda_o} + \frac{\lambda}{\lambda_o^3}\right]/\left(1 + \frac{x}{a}\right)^2 \tag{f}$$

and the system is conservative.

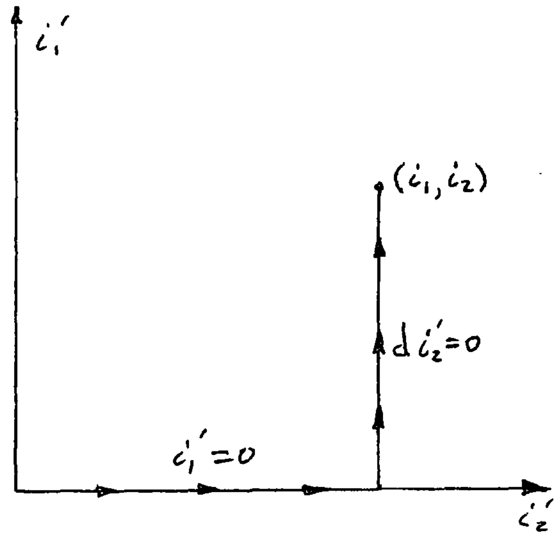
Part b

The stored energy is

$$W_e = \int i d\lambda = \frac{I_o}{[1 + \frac{x}{a}]}\left[\frac{1}{2}\frac{\lambda^2}{\lambda_o} + \frac{1}{4}\frac{\lambda^4}{\lambda_o^3}\right] \tag{g}$$

PROBLEM 3.7

To find the co-energy from the electrical terminal relations alone, we must assume that in the absence of electrical excitations there is no force of electrical origin. Then, the system can be assembled mechanically, with the currents constrained to zero, and there will be no contribution of co-energy in the process (see



Sec. 3.1.1). The co-energy input through the electrical terminal pairs with the mechanical system held fixed is

$$W_m' = \int \lambda_1 di_1 + \lambda_2 di_2 \tag{a}$$

For the path shown in the (i_1, i_2) plane of the figure, this becomes

$$W_m' = \int_o^{i_2} \lambda_2(0, i_2')di_2' + \int_o^{i_1} \lambda_1(i_1', i_2)di_1' \tag{b}$$

amd in view of the given terminal relations, the required co-energy is

$$W_m' = \frac{c}{4} x_2 i_2^4 + b x_1 x_2 i_2 i_1 + \frac{a}{4} x_1 i_1^4 \tag{c}$$

PROBLEM 3.8

Steps (a) and (b) establish the flux in the rotor winding.

$$\lambda_2 = I_o L_m \tag{a}$$

With the current constrained on the stator coil, as in step (c), the current i_1 is known, and since the flux λ_2 is also known, we can use the second terminal equations to solve for the current in the rotor winding as a function of the angular position

$$i_2 = \frac{L_m}{L_2} [I_o - I(t)\cos\theta] \tag{b}$$

This is the electrical equation of motion for the system. To complete the picture, the torque equation must be found. From the terminal relations, the co-energy is

PROBLEM 3.8 (Continued)

$$W'_m = \int \lambda_1 di_1 + \lambda_2 di_2 = \frac{1}{2} L_1 i_1^2 + i_1 i_2 L_m \cos\theta + \frac{1}{2} i_2^2 L_2 \quad \text{(c)}$$

and hence, the electrical torque is

$$T^e = \frac{\partial W'_m}{\partial\theta} = -i_1 i_2 L_m \sin\theta \quad \text{(d)}$$

Now, we use this expression in the torque equation, with i_2 given by (b) and $i_1 = I(t)$

$$\frac{Jd^2\theta}{dt^2} = -\frac{IL_m^2}{L_2}(I_o - I(t)\cos\theta)\sin\theta \quad \text{(e)}$$

This is the required equation of motion. Note that we did not substitute i_2 from (b) into the co-energy expression and then take the derivative with respect to θ. This gives the wrong answer because we have assumed in using the basic energy method to find the torque that i_1,i_2 and θ are thermodynamically independent variables.

PROBLEM 3.9

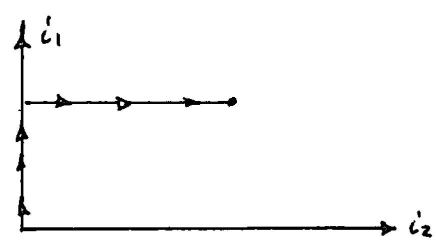
Part a

From the terminal relations, the electrical co-energy is (Table 3.1.1)

$$W'_m = \int \lambda_1 di_1 + \lambda_2 di_2 \quad \text{(a)}$$

or

$$W'_m = \frac{1}{4} ax_1^2 i_1^4 + bx_2^2 x_1 i_1 i_2 + \frac{1}{4} cx_2^2 i_2^4 \quad \text{(b)}$$



Part b

It follows that the required forces are

$$f_1^e = \frac{\partial W'_m}{\partial x_1} = \frac{1}{2} ax_1 i_1^4 + bx_2^2 i_1 i_2 \quad \text{(c)}$$

$$f_2^e = \frac{\partial W'_m}{\partial x_2} = 2bx_2 x_1 i_1 i_2 + \frac{1}{2} cx_2 i_2^4 \quad \text{(d)}$$

PROBLEM 3.9 (Continued)

Part c

There are four equations of motion in the dependent variables i_1, i_2, x_1 and x_2: two of these are the electrical voltage equations, which in view of the terminal equations for the λ's, are

$$-i_1R_1 = \frac{d}{dt}(ax_1^2i_1^3 + bx_2^2x_1i_2) \tag{e}$$

$$v_2(t)-i_2R_2 = \frac{d}{dt}(bx_2^2x_1i_1 + cx_2^2i_2^3) \tag{f}$$

and two are the mechanical force equations

$$0 = \frac{1}{2}\,ax_1i_1^4 + bx_2^2i_1i_2 - Kx_1 \tag{g}$$

$$0 = 2bx_2x_1i_1i_2 + \frac{1}{2}\,cx_2i_2^4 - B\,\frac{dx_2}{dt} \tag{h}$$

PROBLEM 3.10

Part a

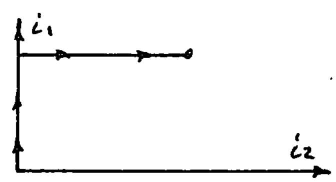
Because the terminal relations are expressed as functions of the current and x, it is most appropriate to use the co-energy to find the force. Hence,

$$W_m' = \int \lambda_1 di_1 + \lambda_2 di_2 \tag{a}$$

which becomes,

$$W_m' = \frac{1}{2}\,L_oi_1^2 + \frac{1}{2}\,Ai_1^2i_2^2x + \frac{1}{2}L_oi_2^2 \tag{b}$$

From this it follows that the force is,



$$f^e = \frac{1}{2}\,Ai_1^2i_2^2 \tag{c}$$

Part b

The currents i_1 and i_2 and x will be used as the dependent variables. Then, the voltage equations for the two electrical circuits can be written, using the electrical terminal equations, as

$$e_1(t) = i_1R_1 + \frac{d}{dt}(L_oi_1 + Ai_1i_2^2x) \tag{d}$$

$$e_2(t) = i_2R_2 + \frac{d}{dt}(Ai_1^2i_2x + L_oi_2) \tag{e}$$

PROBLEM 3.10 (Continued)

The equation for mechanical equilibrium of the mass M is the third equation of motion

$$M\frac{d^2x}{dt^2} = -K(x-x_o) + \frac{1}{2}Ai_1^2i_2^2 \tag{f}$$

PROBLEM 3.11

Part a

The electrical torques are simply found by taking the appropriate derivatives of the co-energy (see Table 3.1.1)

$$T_1^e = \frac{\partial W_m'}{\partial\theta} = -M\sin\theta\cos\psi\ i_1i_2 \tag{a}$$

$$T_2^e = \frac{\partial W_m'}{\partial\psi} = -M\cos\theta\sin\psi\ i_1i_2 \tag{b}$$

Part b

The only torques acting on the rotors are due to the fields. In view of the above expressions the mechanical equations of motion, written using θ,ψ, i_1 and i_2 as dependent variables, are

$$J\frac{d^2\theta}{dt^2} = -M\sin\theta\cos\psi\ i_1i_2 \tag{c}$$

$$J\frac{d^2\psi}{dt^2} = -M\cos\theta\sin\psi\ i_1i_2 \tag{d}$$

Remember that the terminal voltages are the time rates of change of the respective fluxes. Hence, we can make use of the terminal equations to write the current node equations for each of the circuits as

$$I_1(t) = C\frac{d^2}{dt^2}(L_1i_1 + Mi_2\cos\theta\cos\psi) + i_1 \tag{e}$$

$$I_2(t) = G\frac{d}{dt}(Mi_1\cos\theta\cos\psi + L_2i_2) + i_2 \tag{f}$$

Thus, we have four equations, two mechanical and two electrical, which involve the dependent variables θ,ψ, i_1 and i_2 and the known driving functions I_1 and I_2.

PROBLEM 3.12

We can approach this problem in two ways. First from conservation of energy,

$$dW_m' = \lambda_1 di_1 + \lambda_2 di_2 + \lambda_3 di_3 \tag{a}$$

and

$$dW_m' = \frac{\partial W_m'}{\partial i_1} di_1 + \frac{\partial W_m'}{\partial i_2} di_2 + \frac{\partial W_m'}{\partial i_3} di_3 \tag{b}$$

Hence,

$$\lambda_1 = \frac{\partial W_m'}{\partial i_1} ; \lambda_2 = \frac{\partial W_m'}{\partial i_2} ; \lambda_3 = \frac{\partial W_m'}{\partial i_3} \tag{c}$$

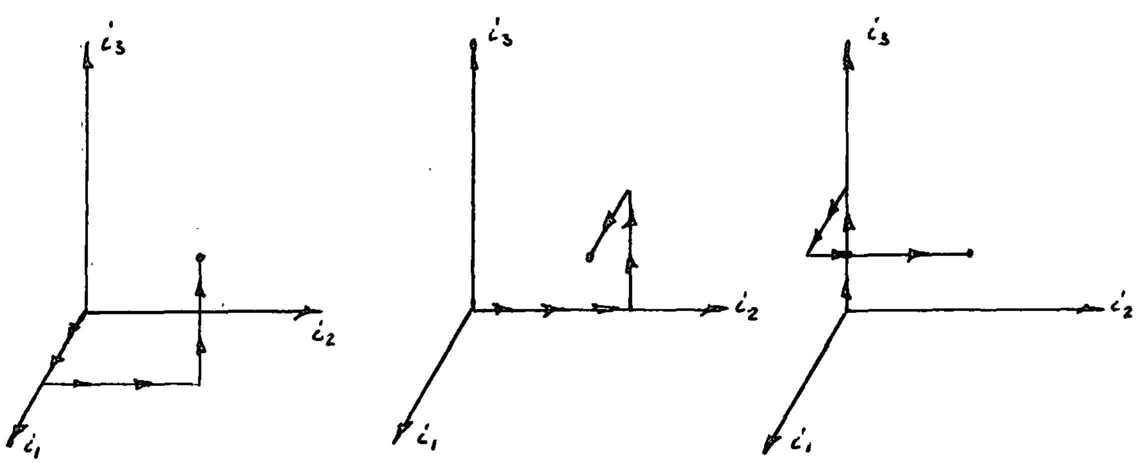
Taking combinations of cross-derivatives, this gives

$$\frac{\partial \lambda_1}{\partial i_2} = \frac{\partial \lambda_2}{\partial i_1} ; \frac{\partial \lambda_2}{\partial i_3} = \frac{\partial \lambda_3}{\partial i_2} ; \frac{\partial \lambda_3}{\partial i_1} = \frac{\partial \lambda_1}{\partial i_3} \tag{d}$$

or

$$L_{12} = L_{21}; L_{23} = L_{32}; L_{31} = L_{13} \tag{e}$$

Another way to show the same thing is to carry out the integrations along the three different paths shown



Since

$$W_m = \int \lambda_1 di_1 + \lambda_2 di_2 + \lambda_3 di_3 \tag{f}$$

PROBLEM 3.12 (Continued)

these paths of integration lead to differing results. For path (a), we have

$$W_m = \frac{1}{2}L_{11}i_1^2 + L_{21}i_1i_2 + \frac{1}{2}L_{22}i_2^2 + L_{31}i_1i_3 + L_{32}i_2i_3 + \frac{1}{2}L_{33}i_3^2 \tag{g}$$

while for path (b)

$$W_m = \frac{1}{2}L_{22}i_2^2 + L_{32}i_2i_3 + \frac{1}{2}L_{33}i_3^2 + \frac{1}{2}L_{11}i_1^2 + L_{12}i_2i_1 + L_{13}i_3i_1 \tag{h}$$

and path (c)

$$W_m = \frac{1}{2}L_{33}i_3^2 + \frac{1}{2}L_{11}i_1^2 + L_{13}i_3i_1 + L_{21}i_1i_2 + \frac{1}{2}L_{22}i_2^2 + L_{23}i_3i_2 \tag{i}$$

These equations will be identical only if (e) holds.

PROBLEM 3.13

Part a

When $\theta = 0$, there is no overlap between the stator and rotor plates, as compared to complete overlap when $\theta = \pi/2$. Because the total exposed area between one pair of stator and rotor plates is $\pi R^2/2$, at an angle θ the area is

$$A = \frac{\pi R^2}{2}\frac{\theta}{(\frac{\pi}{2})} = \theta R^2 \tag{a}$$

There are 2N-1 pairs of such surfaces, and hence the total capacitance is

$$C = (2N-1)\theta R^2\varepsilon_o/g \tag{b}$$

The required terminal relation is then $q = Cv$.

Part b

The system is electrically linear. Hence, $W_e' = \frac{1}{2}Cv^2$ and

$$T^e = \frac{\partial W_e'}{\partial\theta} = \frac{(2N-1)R^2\varepsilon_o v^2}{2g} \tag{c}$$

Part c

There are three torques acting on the shaft, one due to the torsional spring, the second from viscous damping and the third the electrical torque.

PROBLEM 3.13 (Continued)

$$J\frac{d^2\theta}{dt^2} = -K(\theta-\alpha) - B\frac{d\theta}{dt} + \frac{1}{2}\frac{v^2(2N-1)R^2\varepsilon_o}{g} \qquad (d)$$

Part d

The voltage circuit equation, in view of the electrical terminal equation is simply

$$V_o(t) = R\frac{d}{dt}\left[\frac{(2N-1)R^2\theta\varepsilon_o v}{g}\right] + v \qquad (e)$$

Part e

When the rotor is in static equilibrium, the derivatives in (d) vanish and we can solve for $\theta-\alpha$,

$$\theta-\alpha = \frac{V_o^2(2N-1)R^2\varepsilon_o}{2gK} \qquad (f)$$

This equation would comprise a theoretical calibration for the voltmeter if effects of fringing fields could be ignored. In practice, the plates are shaped so as to somewhat offset the square law dependence of the deflections.

PROBLEM 3.14

Part a

Fringing fields are ignored near the ends of the metal coaxial cylinders. In the region between the cylinders, the electric field has the form $\bar{E} = A\bar{i}_r/r$, where r is the radial distance from the axis and A is a constant determined by the voltage. This solution is both divergence and curl free, and hence satisfies the basic electric field equations (See Table 1.2) everywhere between the cylinders. The boundary conditions on the surfaces of the dielectric slab are also satisfied because there is no normal electric field at a dielectric interface and the tangential electric fields are continuous. To determine the constant A, note that

$$\int_a^b E_r dr = -v = A\ln\left(\frac{b}{a}\right); \; A = -v/\ln\left(\frac{b}{a}\right) \qquad (a)$$

The surface charge on the inner surface of the outer cylinder in the regions adjacent to free space is then

PROBLEM 3.14 (Continued)

$$\sigma_f = \frac{v\varepsilon_o}{\ln(\frac{b}{a})b} \tag{b}$$

while that adjacent to regions occupied by the dielectric is

$$\sigma_f = \frac{v\varepsilon}{\ln(\frac{b}{a})b} \tag{c}$$

It follows that the total charge on the outer cylinder is

$$q = v\,\frac{\pi}{\ln(\frac{b}{a})}\,[L(\varepsilon_o+\varepsilon)-x(\varepsilon-\varepsilon_o)] \tag{d}$$

Part b

Conservation of power requires

$$v\,\frac{dq}{dt} = \frac{dW_e}{dt} + f_e\,\frac{dx}{dt} \tag{e}$$

Parts c and d

It follows from integration of (c) that

$$W_e = \frac{1}{2}\frac{q^2}{C} \quad \text{or} \quad W_e' = \frac{1}{2}Cv^2 \tag{f}$$

where

$$C = \frac{\pi}{\ln(\frac{b}{a})}\,[L(\varepsilon_o+\varepsilon)-x(\varepsilon-\varepsilon_o)]$$

Part e

The force of electrical origin is therefore

$$f^e = \frac{\partial W_e'}{\partial x} = -\frac{1}{2}v^2\,\frac{\pi}{\ln(\frac{b}{a})}\,(\varepsilon-\varepsilon_o) \tag{g}$$

Part f

The electrical constraints of the system have been left unspecified. The mechanical equation of motion, in terms of the terminal voltage v, is

$$M\,\frac{d^2x}{dt^2} = -K(x-\ell)-\frac{1}{2}\,\frac{v^2\pi}{\ln(\frac{b}{a})}\,(\varepsilon-\varepsilon_o) \tag{h}$$

PROBLEM 3.14 (Continued)

Part g

In static equilibrium, the inertial term makes no contribution, and (h) can be simply solved for the equilibrium position x.

$$x=\ell-\frac{1}{2}\frac{V_o^2\pi(\varepsilon-\varepsilon_o)}{K\ln(\frac{b}{a})} \tag{i}$$

PROBLEM 3.15

Part a

Call r the radial distance from the origin 0. Then, the field in the gap to the right is, (from Ampere's law integrated across the gaps at a radius r

$$H_\theta^r = Ni/(\beta-\alpha-\theta)r \qquad \text{(directed to the right)} \tag{a}$$

and to the left

$$H_\theta^\ell = Ni/(\beta-\alpha+\theta)r \qquad \text{(directed to the left)} \tag{b}$$

These fields satisfy the conditions that $\nabla\times\bar{H}=0$ and $\nabla\cdot\bar{B}=0$ in the gaps. The flux is computed by integrating the flux density over the two gaps and multiplying by N

$$\lambda = \mu_o DN\int_a^b (H_\theta^\ell + H_\theta^r)dr \tag{c}$$

which, in view of (a) and (b) becomes,

$$\lambda = Li,\ L = \mu_o DN^2\ln(\frac{b}{a})[\frac{1}{\beta-\alpha+\theta}+\frac{1}{\beta-\alpha-\theta}] \tag{d}$$

Part b

The system is electrically linear, and hence the co-energy is simply (See Sec. 3.1.2b)

$$W_m' = \frac{1}{2}Li^2 \tag{e}$$

Part c

The torque follows from (e) as

PROBLEM 3.15 (Continued)

$$T^e = -\frac{1}{2}\mu_o DN^2 \ln(\frac{b}{a})[\frac{1}{(\beta-\alpha+\theta)^2} - \frac{1}{(\beta-\alpha-\theta)^2}]i^2 \qquad (f)$$

Part d

The torque equation is then

$$J\frac{d^2\theta}{dt^2} = -K\theta + T^e \qquad (g)$$

Part e

This equation is satisfied if θ=0, and hence it is possible for the wedge to be in static equilibrium at this position.

PROBLEM 3.16

We ignore fringing fields. Then the electric field is completely between the center plate and the outer plates, where it has the value E = v/b. The constraints on the electrical terminals further require that $v = V_o - Ax$.

The surface charge on the outer plates is $\varepsilon_o v/b$ and hence the total charge q on these plates is

$$q = 2(a-x)\frac{d\varepsilon_o}{b}v \qquad (a)$$

It follows that the co-energy is

$$W_e' = (a-x)\frac{d\varepsilon_o}{b}v^2 \qquad (b)$$

and the electrical force is

$$f^e = \frac{\partial W_e'}{\partial x} = -\frac{d\varepsilon_o}{b}v^2 \qquad (c)$$

Finally, we use the electrical circuit conditions to write

$$f^e = -\frac{d\varepsilon_o}{b}(V_o - Ax)^2 \qquad (d)$$

The major point to be made in this situation is this. One might substitute the voltage, as it depends on x, into (b) before taking the derivative. This clearly gives an answer not in agreement with (d). We have assumed in writing (c) that the variables (v,x) remain thermodynamically independent until after the force has been found. Of course, in the actual situation, external constraints

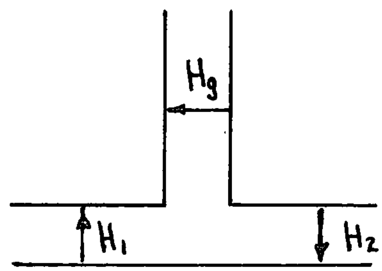
PROBLEM 3.16 (Continued)

relate these variables, but these constraints can only be introduced with care in the energy functions. To be safe they should not be introduced until after the force has been found.

PROBLEM 3.17

Part a

The magnetic field intensities in the gaps can be found by using Ampere's law integrated around closed contours passing through the gaps. These give



$$H_g = N(i_1 + i_2)/g \tag{a}$$

$$H_1 = Ni_1/d \tag{b}$$

$$H_2 = Ni_2/d \tag{c}$$

In the magnetic material, the flux densities are

$$B_1 = \frac{\alpha N^3 i_1^3}{d^3} + \frac{\mu_o N i_1}{d} \tag{d}$$

$$B_2 = \frac{\alpha N^3 i_2^3}{d^3} + \frac{\mu_o N i_2}{d} \tag{e}$$

The flux linking the individual coils can now be computed as simply the flux through the appropriate gaps. For example, the flux λ_1 is

$$\lambda_1 = ND[\ell\mu_o H_g + x\mu_o H_1 + (\ell - x)B_1] \tag{f}$$

which upon substitution from the above equations becomes the first terminal relation. The second is obtained in a similar manner.

Part b

The co-energy is found by integrating, first on i_1 with $i_2 = 0$ and then on i_2 with i_1 fixed at its final value. Hence,

PROBLEM 3.17 (Continued)

$$W'_m = \int \lambda_1 di_1 + \lambda_2 di_2 \tag{g}$$

$$= \frac{1}{2} L_o(1 + \frac{d}{g})i_1^2 + \frac{1}{4} L_o\beta(1 - \frac{x}{\ell})i_1^4 + L_o \frac{d}{g} i_1 i_2$$

$$+ \frac{1}{4} L_o\beta\left(\frac{x}{\ell}\right)i_2^4 + \frac{1}{2} L_o(1 + \frac{d}{g})i_2^2$$

$\left(\frac{x-g}{\ell}\right)$

Part c

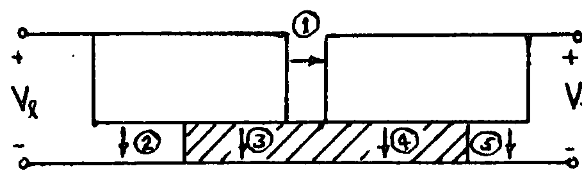
The force of electrical origin follows from the co-energy functions as,

$$f^e = -\frac{1}{4} L_o \frac{\beta}{\ell} i_1^4 + \frac{1}{4} \frac{L_o\beta}{\ell} i_2^4 \tag{h}$$

PROBLEM 3.18

Part a

Assuming simple uniform E fields in the gaps



$E_1 = (V_\ell - V_r)/g$; $E_2 = V_\ell/d = E_3$

$E_4 = E_5 = V_r/d$

These fields leave surface charge densities on the top electrodes

$$\sigma_1 = \varepsilon_o(V_\ell - V_r)/g, \quad \sigma_2 = \varepsilon_o V_\ell/d$$

$$\sigma_3 = [\alpha(V_\ell/d)^2 + \varepsilon_o](V_\ell/d)$$

$$\sigma_4 = [\alpha(V_r/d)^2 + \varepsilon_o](V_r/d)$$

$$\sigma_5 = \varepsilon_o(V_r/d)$$

These surface charge densities cause net charges on the electrodes of

$$q_\ell = \frac{\varepsilon_o wb}{g}(V_\ell - V_r) + \frac{\varepsilon_o wLV_\ell}{d} + \alpha w(L-x)\left(\frac{V_\ell}{d}\right)^3$$

$$q_r = \frac{\varepsilon_o wb}{g}(V_r - V_\ell) + \frac{\varepsilon_o wL}{d} V_r + \alpha w(x-g)\left(\frac{V_r}{d}\right)^3$$

PROBLEM 3.18 (Continued)

Part b

$$W' = \int_{o}^{q_1} q_\ell dV_\ell + \int_{o}^{q_2} q_r dV_r$$
$$\qquad q_2=0 \qquad\qquad q_1=q_1$$

$$= \varepsilon_o w(\frac{b}{g} + \frac{L}{d}) \frac{V_\ell^2}{2} + \frac{\alpha w(L-x)d}{4} (\frac{V_\ell}{d})^4$$

$$+ \varepsilon_o w(\frac{b}{g} + \frac{L}{d}) \frac{V_r^2}{2} - \frac{\varepsilon_o w b}{g} V_\ell V_r + \frac{\alpha w(x-g)d}{4} (\frac{V_r}{d})^4$$

$$f_e = \frac{\partial W'}{\partial x} = \frac{\alpha wd}{4} [(\frac{V_r}{d})^4 - (\frac{V_\ell}{d})^4] \qquad \text{(pulled to side with more voltage)}$$

PROBLEM 3.19

Part a

The rotating plate forms a simple capacitor plate with respect to the other two curved plates. There is no mutual capacitance if the fringing fields are ignored. For example, the terminal relations over the first half cycle of the rotor are

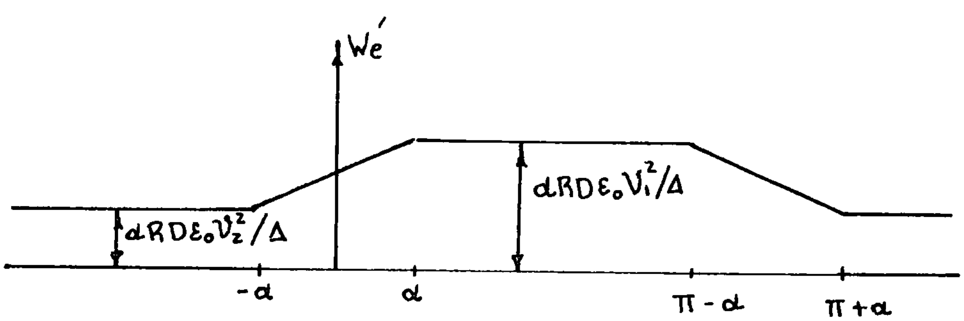
$$-\alpha<\theta<\alpha; \; q_1 = \frac{(\alpha+\theta)RD\varepsilon_o v_1}{\Delta} \; ; \; q_2 = \frac{(\alpha-\theta)RD\varepsilon_o v_2}{\Delta} \qquad (a)$$

$$\alpha<\theta<\pi-\alpha; \; q_1 = \frac{2\alpha RD\varepsilon_o v_1}{\Delta} \; ; \; q_2 = 0 \qquad (b)$$

So that the co-energy can be simply written as the sum of the capacitances for the two outer electrodes relative to the rotor.

$$W_e' = \frac{1}{2} C_1 v_1^2 + \frac{1}{2} C_2 v_2^2 \qquad (c)$$

The dependence of this quantity on θ is as shown below

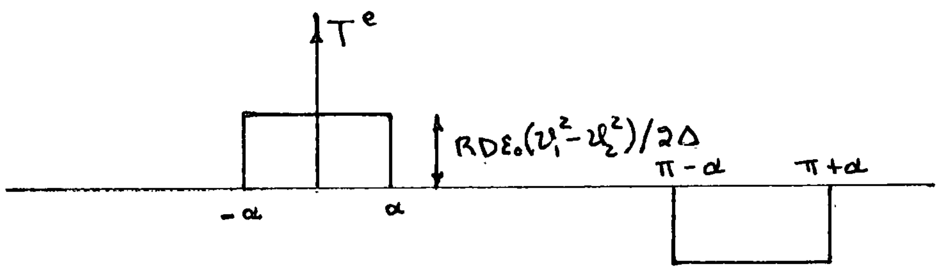


(d)

PROBLEM 3.19 (Continued)

Part b

The torque is the spatial derivative of the above function



Part c

The torque equation is then

$$J\,\frac{d^2\theta}{dt^2} = T^e$$

where T^e is graphically as above.

PROBLEM 3.20

Part a

The electric energy is

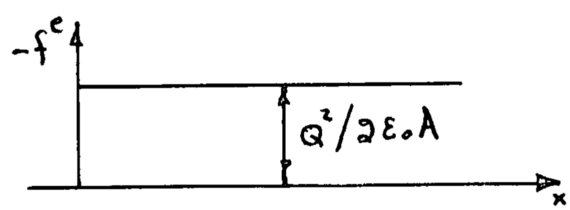
$$W_e = \frac{1}{2}\,q^2/C \qquad \text{(a)}$$

where

$$C = \varepsilon A/d(1 + \frac{\varepsilon x}{\varepsilon_o d}) \qquad \text{(b)}$$

It follows that the force on the upper plate due to the electric field is,

$$-f=f^e=-\frac{\partial W_e}{\partial x}=-\frac{1}{2}\frac{Q^2}{\varepsilon_o A}$$



So long as the charge on the plate is constant, so also is the force.

PROBLEM 3.20 (Continued)

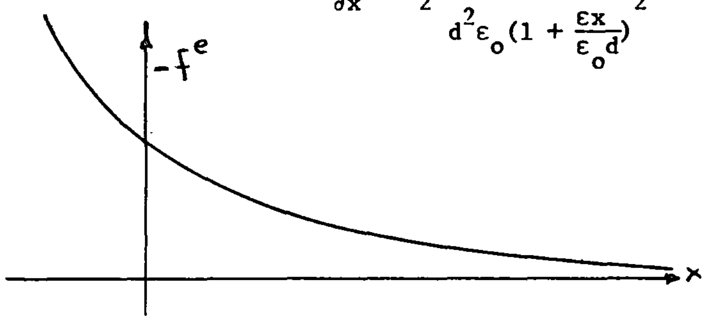
Part b

The electric co-energy is

$$W_e' = \frac{1}{2} Cv^2 \tag{c}$$

and hence the force, in terms of the voltage is

$$-f=f^e = \frac{\partial W_e'}{\partial x} = \frac{1}{2} \frac{V^2\varepsilon^2 A}{d^2\varepsilon_o(1+\frac{\varepsilon x}{\varepsilon_o d})^2} \tag{d}$$



The energy converted to mechanical form is $\int f^e\,dx$. The contribution to this integral from d→c and b→a in the figure is zero. Hence,

$$\text{Energy converted to mechanical form} = \int_{\varepsilon_o d/\varepsilon}^{2\varepsilon_o d/\varepsilon} f^e(2Q_o,x)\,dx$$

$$+\int_{2\varepsilon_o d/\varepsilon}^{\varepsilon_o d/\varepsilon} f^e(Q_o,x)\,dx = -\frac{3}{2}\frac{dQ_o^2}{A\varepsilon} \tag{e}$$

That is, the energy $3dQ_o^2/2A\varepsilon$ is converted from mechanical to electrical form.

PROBLEM 3.21

Part a

The magnetic energy stored in the coupling is

$$W_m = \frac{1}{2}\lambda^2/L \tag{a}$$

where $L = L_o/(1+\frac{x}{a})$

Hence, in terms of λ, the force of electrical origin is

$$-f=f^e = -\frac{\partial W_m}{\partial x} = -\lambda^2/2aL_o \tag{b}$$

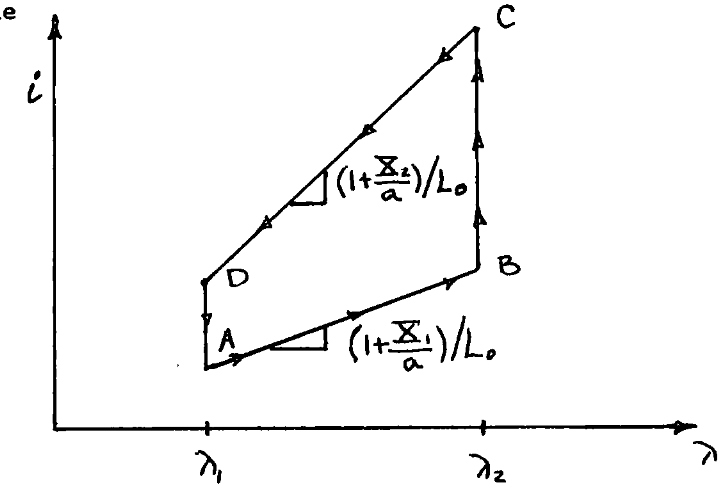
PROBLEM 3.21 (Continued)

Part b

According to the terminal equation, i depends on (λ,x) according to

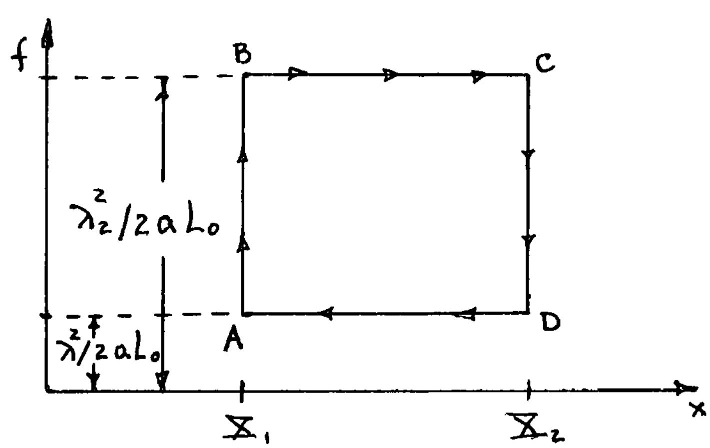
$$i = \frac{\lambda}{L_o}\left(1 + \frac{x}{a}\right) \tag{c}$$

Thus, the process represented in the λ-x plane has the corresponding path in the i-λ plane



Path c

At the same time, the force traverses a loop in the f-x plane which, from (b) is,



PROBLEM 3.21 (Continued)

Part d

The energy converted per cycle to mechanical form is $\int f^e dx$. Hence,

$$\text{Energy converted to mechanical form} = \int_B^C f^e dx + \int_D^A f^e dx \tag{d}$$

$$= -(\lambda_2^2-\lambda_1^2)(X_2-X_1)/2aL_o \tag{e}$$

That is, the energy converted to electrical form per cycle is $(\lambda_2^2-\lambda_1^2)(X_2-X_1)/2aL_o$. (Note that the energy stored in the coupling, summed around the closed path, is zero because the coupling is conservative.)

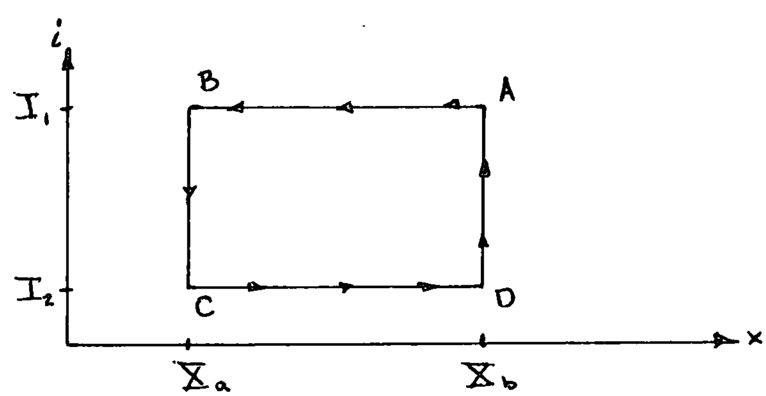
PROBLEM 3.22

Part a

The plates are pushed apart by the fields. Therefore energy is converted from mechanical form to either electrical form or energy storage in the coupling as the plate is moved from X_b to X_a. To make the net conversion from mechanical to electrical form, we therefore make the current the largest during this phase of the cycle or, $I_1>I_2$.

Part b

With the currents related as in part a, the cycle appears in the i-x plane as shown



PROBLEM 3.22 (Continued)

Quantitatively, the magnetic field intensity into the paper is $H = I/D$ so that $\lambda = \mu_o Ixh/D$. Hence,

$$W_m' = \frac{1}{2}\left(\frac{\mu_o xh}{D}\right)I^2 \tag{a}$$

and

$$f^e = \frac{\partial W_m'}{\partial x} = \frac{1}{2}\left(\frac{\mu_o h}{D}\right)I^2 \tag{b}$$

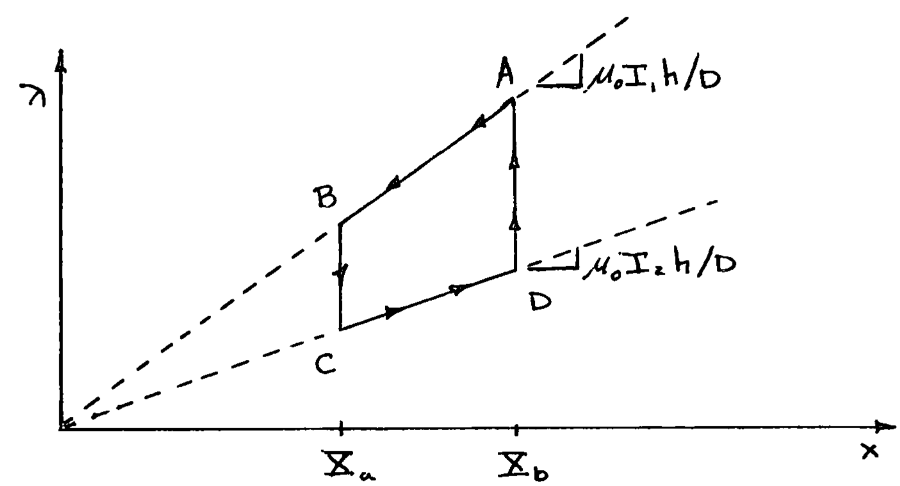
Because the cycle is closed, there is no net energy stored in the coupling, and the energy converted to electrical form is simply that put in in mechanical form:

$$\text{Mechanical to electrical energy per cycle} = -\int_A^B f^e dx - \int_C^D f^e dx \tag{c}$$

$$= \frac{1}{2}\frac{\mu_o h}{D}(X_b - X_a)(I_1^2 - I_2^2) \tag{d}$$

Part c

From the terminal equation and the defined cycle conditions, the cycle in the λ-x plane can be pictured as



The energy converted to electrical form on each of the legs is

PROBLEM 3.22 (Continued)

$$(A\rightarrow B) \qquad -\int I_1 d\lambda = -\int_{\mu_o I_1 X_b h/D}^{\mu_o I_1 X_a h/D} I_1 d\lambda = -\frac{\mu_o I_1^2 h}{D}(X_a - X_b) \qquad \text{(e)}$$

$$(B\rightarrow C) \qquad -\int i d\lambda = -\int_{\mu_o I_1 X_a h/D}^{\mu_o I_2 X_a h/D} \frac{\lambda D d\lambda}{\mu_o X_a h} = -\frac{\mu_o X_a h}{D}(I_2^2 - I_1^2) \qquad \text{(f)}$$

$$(C\rightarrow D) \qquad -\int \dot{I}_2 d\lambda = \frac{\mu_o I_2^2 h}{D}(X_a - X_b) \qquad \text{(g)}$$

$$(D\rightarrow A) \qquad -\int i d\lambda = \frac{\mu_o X_b h}{2D}(I_2^2 - I_1^2) \qquad \text{(h)}$$

The sum of these is equal to (c). Note however that the mechanical energy input on each leg is not necessarily converted to electrical form, but can be stored in the coupling.