PROBLEMS

11.1. In Fig. 11P.1 a static elastic material is constrained along its vertical sides so that

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0.$$

In the absence of a gravitational field, the material has surfaces at $x_1 = 0$ and $x_1 = L$.

(a) Compute the material displacement $\delta_1(x_1)$ caused by the gravitational field.

(b) Find all components of the stress T_{ij}.

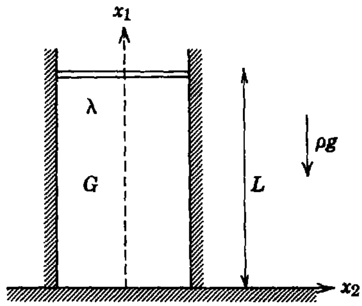

Fig. 11P.1

11.2. In Fig. 11P.2 a slab of elastic solid with constants ρ, G, λ and a thickness L is attached on one side to a rigid wall at $x_1 = 0$. A perfectly conducting thin plate of mass M is attached to the other side of the solid. A second perfectly conducting plate is fixed at $x_1 = -L - d$. Assume that $\partial/\partial x_2 = \partial/\partial x_3 = 0$ and $\delta_1(-L, t) \ll d$.

(a) If the voltage between the two capacitor plates is $V(t) = V_0 + V_1 \cos \omega t$, find $\delta_1(-L, t)$ $V_1 \ll V_0$.

(b) For what frequency range does the *mechanical* part of the system appear lumped?

(c) Give the mechanical lumped parameters for the frequency range defined in (b).

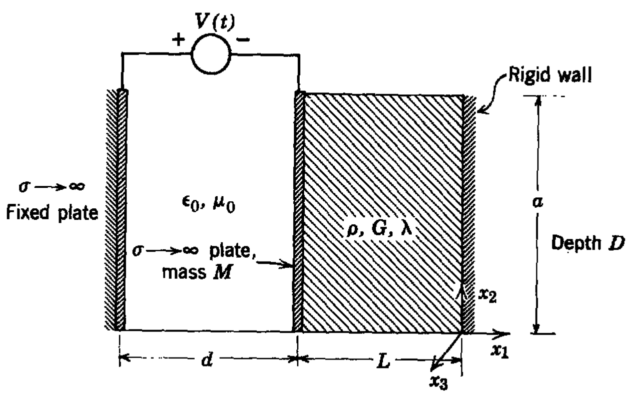

Fig. 11P.2

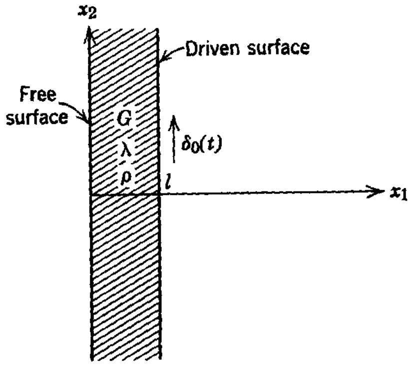

Fig. 11P.3

11.3. A slab of elastic material of length l in the x_1-direction and infinite in extent in the x_2- and x_3-directions has the elastic constants G and λ and mass density ρ. Its surface at $x_1 = l$ is driven in the x_2-direction uniformly by a displacement source $\delta_0(t)$. The surface at $x_1 = 0$ is free to move in the x_2-direction without restraint.

Assume that $\delta_0(t) = \text{Re}\,(\hat{\delta}_0 e^{j\omega t})$, where $\hat{\delta}_0$ and ω are given constants. Neglect the force of gravity and assume that

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0.$$

(a) Find the stress and displacement in the slab.
(b) In the limit of low frequency to what lumped mechanical element does the slab correspond?
(c) Find the lowest frequency for which the slab may be said to "resonate."

11.4. In a coordinate system (x_1, x_2, x_3) a surface with the normal vector $\mathbf{n}$ and supporting the stress T_{ij} is subject to the traction (see Section 8.2.2)* $\tau_i = T_{ij}n_j$. Assume that the stress components T_{ij} are known and that there is a surface with an orientation such that the traction is in the same direction as the normal vector; that is, $\tau_i = \alpha\delta_{ij}n_j$, where α is the stress acting normal to the surface.

(a) Write three equations in the three unknowns (n_1, n_2, n_3).
(b) Because these equations are homogeneous, their compatibility requires that the determinant of the coefficients vanish. Show that this gives an expression for α.
(c) Consider the case in which $T_{12} = T_{21} = T_0$ and all other components are zero. What are the possible values of the normal stress α? Compare your result with that found in Example 11.2.1.

11.5. In Example 11.2.1 it was shown that the three elastic constants (G, E, ν) must be related if a perfectly elastic material is isotropic [(g) of that example]. This was done by considering the transformation of a particular case of stress and strain from one coordinate system to a second with the same x_3-axis but a 45° rotation in the x_1-x_2 plane. Follow the arguments presented in Example 11.2.1 to show that the relation is implied for an arbitrary stress condition and an arbitrary rotation of coordinates. Remember that the a_{ij} that determine the rotation of coordinates are related by (8.2.23)* and that if $T'_{pq} = a_{pr}a_{qs}T_{rs}$ then $T_{rs} = a_{pr}a_{qs}T'_{pq}$.

* Appendix G.

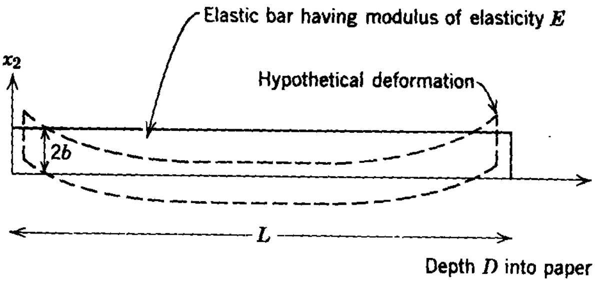

Fig. 11P.6

11.6. An elastic bar is often used in musical instruments as a source of audio-frequency tone. The bar is suspended by strings, attached to it at such points that the transverse (x_2) motion of the elastic material is not appreciably inhibited. (Examples are the vibraharp and marimba.) If the bar is struck by a mallet, it vibrates at one or more of its resonance frequencies. We consider here the problem of finding these frequencies, under the assumption that the bar is as shown in Fig. 11P.6. The bar is supported so that transverse motions are uninhibited, that is, both ends are free.

(a) Find an equation of the form $\cos\beta\cosh\beta = 1$ [$\beta = \beta(\omega)$] which stipulates the resonance frequencies.
(b) Use a graphical solution of the equation found in (a) to determine the two lowest resonance frequencies in terms of E and the dimensions of the bar.
(c) Sketch the transverse deflection as a function of x_1 for the lowest nontrivial mode.

11.7. A thin elastic beam of thickness $2b$, density ρ, and modulus of elasticity E is clamped on both ends to rigid walls. The total length of the beam is L, as shown in Fig. 11P.7.

(a) If the beam is suddenly struck from above, what is the lowest (nonzero) frequency at which it will "ring"; that is, what is its lowest natural frequency?
(b) Give a numerical answer for (a) in Hertz if the beam is steel with length $L = 50$ cm and thickness $2b = 0.10$ cm.
(c) What is the numerical value, again in Hertz, of the next higher resonance frequency?

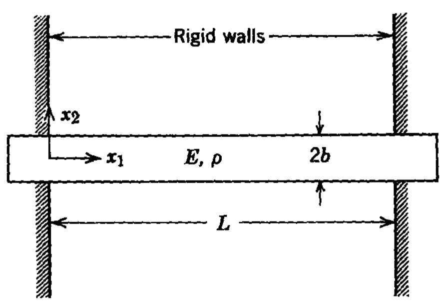

Fig. 11P.7

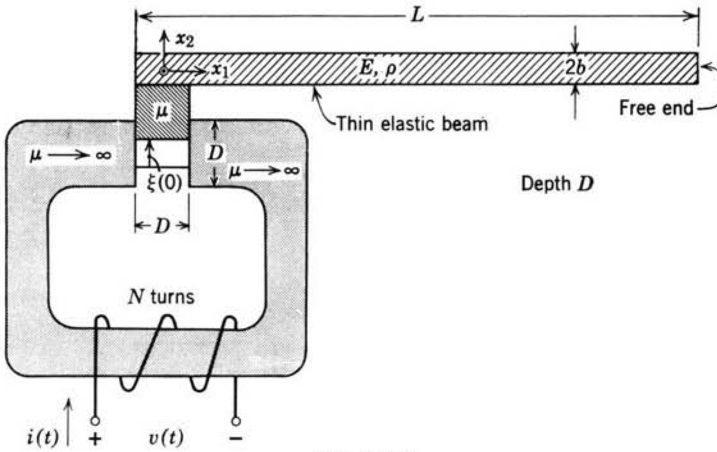

Fig. 11P.8

11.8. The electromechanical system shown in Fig. 11P.8 consists of a long thin elastic beam attached to the plunger of an electromagnet. The plunger has permeability μ and is free to slide between the faces of the electromagnet. Treat the plunger as a rigid body with mass M. Assume that $D \ll L$. The coil on the electromagnet is now excited with a current $i(t) = I_0 + i_1 \cos \omega t$, where $|i_1| \ll I_0$. You may assume that an externally applied force F_0 holds the plunger in equilibrium against the current I_0. Also in equilibrium, the displacement of the beam $\xi(0) = 0$.

(a) What is the value of F_0 required for equilibrium?

(b) Find an expression for the electrical impedance $Z(j\omega)$ seen at the terminals of the coil, where $Z(j\omega) = \hat{v}(j\omega)/i_1$, and $\hat{v}(j\omega)$ is the complex amplitude of the steady-state voltage developed at the terminals.

(c) What is the expression that determines the poles of the impedance $Z(j\omega)$?

11.9. A thin beam clamped to two rigid walls is shown in Fig. 11P.9. Suppose that the beam is perfectly conducting and that it is placed between two perfectly conducting rigid

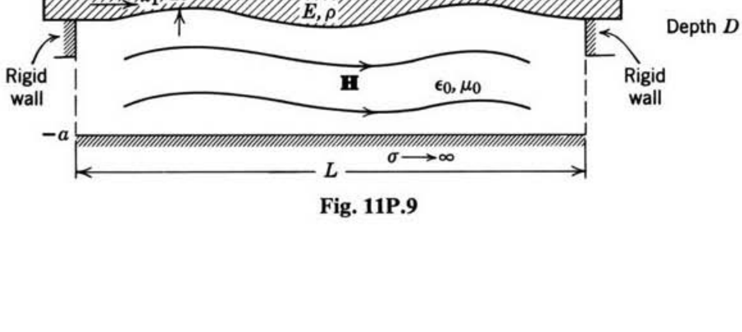

Fig. 11P.9

plates at $x_2 = \pm a$. Assume that there is a magnetic field trapped between the beam and the plates, so that *when the beam is flat* $\mathbf{H} = H_0\mathbf{i}_1$ on both sides of the beam. (In the perfect conductor $\mathbf{H} = 0$.) Make the approximations that wavelengths of a disturbance on the beam are *long* compared with a and that the magnetic field is always uniform in the x_2- and x_3-directions.

(a) Write the equation of motion for the beam.
(b) Compute the first resonance frequency of the beam.
(c) Compare the result of (b) with Problem 11.7 and give a physical explanation for any differences which occur.
(d) Can the system be unstable? Explain.

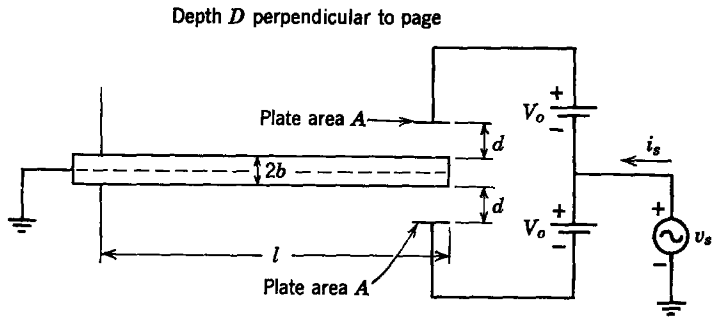

Fig. 11P.10

11.10. The system shown schematically in Fig. 11P.10 is similar to that discussed in Section 11.4.2b. The material of the beam is steel and the system constants and dimensions are

$$E = 2.2 \times 10^{11}\ \text{N/m}^2 \qquad A = 10^{-4}\ \text{m}^2$$
$$\rho = 7.9 \times 10^{3}\ \text{kg/m}^3 \qquad D = 10^{-2}\ \text{m}$$
$$l = 10^{-1}\ \text{m} \qquad b = 10^{-3}\ \text{m}$$
$$V_o = 1000\ \text{V} \qquad d = 10^{-3}\ \text{m}$$

We are interested in investigating the impedance seen by the signal source v_s for values of exciting frequency near the first resonance of the elastic bar. This type of information would be essential if we planned to use this system to control the frequency of an oscillator. For sinusoidal excitation $v_s = \text{Re}\,[\hat{v}_s e^{j\omega t}]$ and small-signal, steady-state operation:

(a) Find a *literal* expression for the input impedance $Z(j\omega) = \hat{v}_s/\hat{\imath}_s$, where $\hat{\imath}_s$ is the complex amplitude of the input current.
(b) For the numerical values given find a *numerical value* for the lowest frequency ω_0 at which the impedance $Z(j\omega)$ has a zero.
(c) Assume operation at frequencies near ω_0 by setting $\omega = \omega_0 + \Delta\omega$, where $|\Delta\omega| \ll \omega_0$ and $\Delta\omega$ can be either positive or negative. For this restriction the impedance Z appears as a series LC circuit. Find numerical values for the equivalent capacitance C and equivalent inductance L.

11.11. Consider the planar elastic waveguide of Fig. 11.4.18 but with the walls at $x_2 = 0$ and $x_2 = d$ fixed.

(a) Find the dispersion equation for waves in the form of

$$\delta_3 = \text{Re}\,\hat{\delta}(x_2)\exp j(\omega t - kx_1).$$

(b) Sketch the results of part (a) as an ω-k plot and compare with Fig. 11.4.19. Is there a principal mode of propagation?

11.12. A cylindrical, circular elastic section of material with the shear modulus G, density ρ, and radius R is embedded in a perfectly rigid solid so that the material at $r = R$ is fixed. This structure is to be used as a waveguide for elastic shear waves. To find the dispersion equation for these waves, we confine interest to material displacements in the form of $\boldsymbol{\delta} = \delta_\theta(r, z, t)\mathbf{i}_\theta$. Find the dispersion equation for all modes in this form. (A discussion of Bessel's functions is given on p. 207 of S. Ramo, J. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*, Wiley, New York, 1965.)

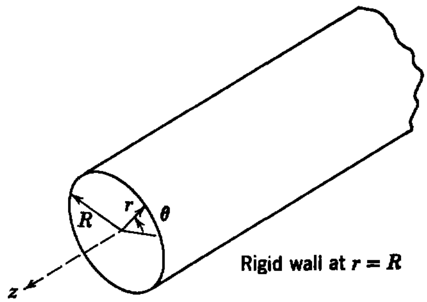

Fig. 11P.12