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SOLUTIONS TO CHAPTER 7

7.1 CONDUCTION CONSTITUTIVE LAWS

7.1.1 If there are as many conduction electrons as there are atoms, then their number density is

$$N_- = \frac{A}{M_o}\rho = \frac{(6.023 \times 10^{26}(8.9 \times 10^3)}{63.5} = 8.4 \times 10^{28}\frac{\text{electrons}}{m^3} \tag{1}$$

The mobility is then

$$\mu_- = \frac{\sigma}{N_- q_-} = \frac{5.8 \times 10^7}{(8.4 \times 10^{38})(1.6 \times 10^{-19})} = 4.3 \times 10^{-3} \tag{2}$$

The electric field required to produce a current density of $1A/\text{cm}^2$ is

$$E = \frac{J}{\sigma} = \frac{10^4}{5.9 \times 10^7} = 1.7 \times 10^{-4} v/m \tag{3}$$

Thus, in copper, the velocity of the electrons giving rise to this current density is only

$$v_- = \mu_- E = (4.3 \times 10^{-3})(1.7 \times 10^{-4}) = 7.4 \times 10^{-7} m/s \tag{4}$$

7.2 STEADY OHMIC CONDUCTION

7.2.1 Boundary conditions on the conducting region are that $\Phi = 0, \Phi = v$ on the perfectly conducting surfaces at $r = a$ and $r = b$ respectively and that there is no normal current density on the insulating surfaces where $z = 0, z = d$. The latter are satisfied by a potential that is independent of the axial coordinate, so an appropriate solution to Laplace's equation, arranged to be zero on the outer electrode, is

$$\Phi = A ln(r/a) \tag{1}$$

The coefficient is adjusted to make the potential v on the inner electrode so that $A = v/ln(b/a)$ and (1) becomes

$$\Phi = v ln(r/a)/ln(b/a) \tag{2}$$

The current density is

$$J_r = \sigma E_r = -\frac{v\sigma}{ln(b/a)}\frac{1}{r} \tag{3}$$

and so the total current is

$$i = 2\pi b d J_r = -\frac{2\pi b d\sigma}{ln(b/a)}\frac{1}{b} = \frac{2\pi\sigma d}{ln(a/b)}v = \frac{v}{R} \tag{4}$$

Thus, R is as given.

7.2.2 The net current passing through the wire connected to the inner spherical electrode, i, must be equal to the net current at any radius r.

$$i = \int_S \mathbf{J} \cdot d\mathbf{a} = 4\pi r^2 \sigma E_r \Rightarrow E_r = \frac{i}{4\pi\sigma r^2} \tag{1}$$

Thus,

$$v = \int_b^a E_r dr = \frac{i}{4\pi\sigma} \int_b^a \frac{dr}{r^2} = \frac{i}{4\pi\sigma}\Big[\frac{1}{b} - \frac{1}{a}\Big] \tag{2}$$

By definition $v = iR$ so $R = \big(\frac{1}{b} - \frac{1}{a}\big)/4\pi\sigma$.

7.2.3 (a) Associated with the uniform field is the potential

$$\Phi = -\frac{v}{d}(y - d) \tag{1}$$

If the surrounding region is insulating relative to that between the electrodes, the normal component of the current density on the conductor surfaces bounded by the insulating surroundings is zero. The potential is constrained on the remainder of the surface enclosing the conductors, so the solution is uniquely specified. Provided the laws are satisfied everywhere inside the conducting region, the solution is exact. The given solution does indeed satisfy the boundary conditions on the surfaces of the conducting region. In the case of (a), the potential and normal component of current density must be continuous across the interior interface. Further, in the uniformly conducting regions of (a), Laplace's equation must be satisfied, as it is by a uniform field. In the case of (b), (7.2.4) is satisfied by the given potential.

(b) The total current is related to v by integrating the current density over the surface of the lower conductor.

$$i = J^a ac + J^b bc = \sigma_a ac\frac{v}{d} + \sigma_b bc\frac{v}{d} = \frac{c(a\sigma_a + b\sigma_b)}{d}v = Gv \tag{2}$$

(c) A similar calculation gives the resistance in the second case.

$$i = c\int_0^l \frac{\sigma v}{d} x dx = \frac{c}{d}\sigma_a v \int_0^l \Big(1 + \frac{x}{l}\Big) dx = \frac{2\sigma_a lc}{2d} v = Gv \tag{3}$$

7.2.4 The potential in each of the uniformly conducting regions takes the form

$$\Phi^a = -A\phi + C; \quad \Phi^b = -B\phi + F \tag{1}$$

where the four coefficients are adjusted to make the potentials zero and v on the respective electrodes, and make both the potential and the normal current density continuous at the interface between the conductors. On the surfaces at $r = a$ and

$r = b$, the current density must be zero, as it is for the potential of (1) because the electric field

$$\mathbf{E} = -\frac{1}{r}\frac{\partial\Phi}{\partial\phi} = \begin{cases} (A/r)\mathbf{i}_\phi \\ (B/r)\mathbf{i}_\phi \end{cases} \tag{2}$$

has no radial component. Rather than proceeding to determine the four coefficients in (1), we work directly with the electric field. The integration of $\mathbf{E}$ from one electrode to the other must be equal to the applied voltage.

$$\frac{\pi}{2}r\frac{A}{r} + \frac{\pi}{2}r\frac{B}{r} = v \tag{3}$$

Further, the current density must be continuous at the interface.

$$\sigma_a\frac{A}{r} = \frac{\sigma_b B}{r} \tag{4}$$

It follows from these relations that

$$A = 2v/\pi(1 + \sigma_a/\sigma_b) \tag{5}$$

The current through any cross-section of the material [say region (a)] must be equal to that through the wire. Thus,

$$i = d\int_b^a \sigma_a E_\phi dr = \left[\frac{2d\sigma_a}{\pi(1+\sigma_a/\sigma_b)}\int_b^a \frac{dr}{r}\right]v \equiv Gv \tag{6}$$

and the resistance is

$$G = \frac{2d\sigma_a}{\pi(1+\frac{\sigma_a}{\sigma_b})}ln(a/b) \tag{7}$$

7.2.5 (a) From (7.2.23)

$$G = A\sigma_o/\int_0^d \left(1+\frac{y}{a}\right)dy = \frac{A\sigma_o}{d\left[1+\frac{1}{2}\frac{d}{a}\right]} \tag{1}$$

(b) We need the electric field, which follows from (7.2.19) by using the result of (1) to evaluate $J_o = i/A = Gv/A$

$$E_y = \frac{J_o}{\sigma} = \frac{G}{A\sigma_o}\left(1+\frac{y}{a}\right)v \tag{2}$$

Thus, the unpaired charge density is evaluated using (7.2.8).

$$\rho_u = -\frac{\epsilon\left(1+\frac{y}{a}\right)}{\sigma_o}\frac{G\left(1+\frac{y}{a}\right)}{A\sigma_o}\frac{d}{dy}\left[\frac{\sigma_o}{\left(1+\frac{y}{a}\right)}\right] = \frac{\epsilon G}{A\sigma_o a} \tag{3}$$

7.2.6 (a) The inhomogeneity in permittivity has no effect on the resistance. It is therefore given by (7.2.25).

(b) With the steady conduction laws stipulating that the electric field is uniform, the unpaired charge density follows from Gauss' law.

$$\rho_u = \nabla \cdot \epsilon \mathbf{E} = \nabla \cdot \left(\epsilon \frac{v}{d} \mathbf{i}_y\right) = \frac{v}{d}\frac{\partial \epsilon}{\partial y} = -\frac{\epsilon_a v}{da}\frac{1}{\left(1+\frac{y}{a}\right)^2} \tag{1}$$

7.2.7 At a radius r, the area of the conductor (and with $r = a$ and $r = b$, of the outer and inner electrodes, respectively) is

$$A = 2\pi r^2[1-\cos(\alpha/2)] \tag{1}$$

Consistent with the insulating surfaces of the conductor is the requirement that the current density and associated electric field be radial. Current conservation (fundamentally, the requirement that the current density be solenoidal) then gives as a solution to the field laws

$$\sigma E_r \left[2\pi r^2 \left(1-\cos\frac{\alpha}{2}\right)\right] = i \tag{2}$$

and it follows that

$$E_r = \frac{ia^2}{2\pi\sigma_o\left(1-\cos\frac{\alpha}{2}\right)}\frac{1}{r^4} \tag{3}$$

The voltage follows as

$$v = \int_b^a E_r dr = i(a^3-b^3)/6\pi\sigma_o\left(1-\cos\frac{\alpha}{2}\right)b^3 a \tag{4}$$

and this relation takes the form $i = vG$, where G is as given.

7.2.8 There can be no current density normal to the interfaces of the conducting material having normals in the azimuthal direction. These boundary conditions are satisfied by an axially symmetric solution in which the current density is purely radial. In that case, both $\mathbf{E}$ and $\mathbf{J}$ are independent of ϕ. Then, the total current is related to the current density and (through Ohm's law) electric field intensity at any radius r by

$$i = 2\pi\alpha d r J_r = 2\pi\alpha d\sigma_o a E_r \tag{1}$$

Thus,

$$E_r = \frac{i}{2\pi\alpha d\sigma_o a} \tag{2}$$

and because

$$\int_b^a E_r dr = \frac{i(a-b)}{2\pi\alpha d\sigma_o a} = v \tag{3}$$

$$G = 2\pi\alpha d a\sigma_o/(a-b) \tag{4}$$

7.3 DISTRIBUTED CURRENT SOURCES AND ASSOCIATED FIELDS

7.3.1 In the conductor, the potential distribution is a particular part comprised of the potential due to the point current soruce, (6) with $i_p \rightarrow I$ and

$$r = \sqrt{x^2 + (y-h)^2 + z^2}$$

In order to satisfy the condition that there be no normal component of $\mathbf{E}$ at the interface, a homogeneous solution is added that amounts to a second source of the same sign in the lower half space. Of course, such a current source could not really exist in the lower region so if the field in the upper region is to be given some equivalent physical situation, it should be pictured as equivalent to a pair of like-signed point current sources in a uniform conductor. In any case, this second source is located at $r = \sqrt{x^2 + (y+h)^2 + z^2}$ and hence the potential in the conductor is as given. In the lower region, the potential must satisfy Laplace's equation everywhere (there are no charges in the lower region). The field in this region is uniquely specified by requiring that the potential be consistent with (a) evaluated at the interface

$$\Phi(x, y=0, z) = \frac{2I}{4\pi\sigma\sqrt{x^2 + h^2 + z^2}} \tag{1}$$

and that it go to zero at infinity in the lower half-space. The potential that matches these conditions is that of a point charge of magnitude $q = 2I\epsilon/\sigma$ located on the y axis at $y = h$, the given potential.

7.3.2 (a) First, what is the potential associated with a uniform line current in a uniform conductor? In the steady state

$$\oint_S \mathbf{J} \cdot d\mathbf{a} = 0 \tag{1}$$

and for a surface S that has radius r from the line current,

$$K_l = 2\pi r J_r = 2\pi r \sigma E_r \Rightarrow E_r = \frac{K_l}{2\pi\sigma r} \tag{2}$$

Within a constant, the associated potential is therefore

$$\Phi = -\frac{K_l}{2\pi\sigma} ln(r) \tag{3}$$

To satisfy the requirement that there be no normal current density in the plane $y = 0$, the potential is that of the line current located at $y = h$ and an image line current of the same polarity located at $y = -h$.

$$\Phi^a = -\frac{K_l}{2\pi\sigma}[ln\sqrt{x^2 + (y-h)^2} + ln\sqrt{x^2 + (y-h)^2}] \tag{4}$$

Note that the normal derivative of this expression in the plane $y = 0$ is indeed zero.

(b) In the lower region, the potential must satisfy Laplace's equation everywhere and match the potential of the conductor in the plane $y = 0$.

$$\Phi^b(y = 0) = -\frac{K_l}{\pi\sigma} ln\sqrt{x^2 + h^2} \tag{5}$$

This has the potential distribution of an image line current located at $y = h$. With the magnitude of this line current adjusted so that the potential of (5) is matched at $x = 0$,

$$\Phi^b = -\frac{K_l}{\pi\sigma} ln\sqrt{x^2 + (y - h)^2} \tag{6}$$

the potential is matched at every other value of x as well.

7.3.3 First, the potential due to a single line current is found from the integral form of (2).

$$2\pi\sigma r E_r = K_l \Rightarrow E_r = \frac{K_l}{2\pi\sigma r} \tag{1}$$

Thus, for a single line current,

$$\Phi = -\frac{K_l}{2\pi\sigma} lnr \tag{2}$$

For the pair of line currents, spaced by the distance d,

$$\Phi = -\frac{K_l}{2\pi\sigma}[ln(r - d\cos\phi) - lnr] = -\frac{K_l}{2\pi\sigma} ln\left[1 - \frac{d\cos\phi}{r}\right] \rightarrow \frac{K_l d\cos\phi}{2\pi\sigma r} \tag{3}$$

7.4 SUPERPOSITION AND UNIQUENESS OF STEADY CONDUCTION SOLUTIONS

7.4.1 (a) At $r = b$, there is no normal current density so that

$$J_r(r = b) = 0 \Rightarrow \frac{\partial\Phi}{\partial r}(r = b) = 0 \tag{1}$$

while at $r = a$,

$$J_r = -J_o\cos\theta = -\sigma\frac{\partial\Phi}{\partial r}(r = a) \tag{2}$$

Because the dependence of the potential must be the same as the radial derivative in (2), assume the solution takes the form

$$\Phi = Ar\cos\theta + B\frac{\cos\theta}{r^2} \tag{3}$$

Substitution into (1) and (2) then gives the pair of equations

$$\begin{bmatrix} 1 & -2b^{-3} \\ \sigma & -2\sigma a^{-3} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ J_o \end{bmatrix} \tag{4}$$

from which it follows that

$$A = J_o/\sigma[1-(b/a)^3]; \quad B = J_o b^3/2\sigma[1-(b/a)^3] \tag{5}$$

Substitution into (3) results in the given potential in the conducting region.

(b) The potential inside the hollow sphere is now specified, because we know that the potential on its wall is

$$\Phi(r=b) = 3J_o b/2\sigma[1-(d/a)^3] \tag{6}$$

Here, the origin is included, so the only potential having the required dependence is

$$\Phi = Cr\cos\theta \tag{7}$$

Determination of C by evaluating (7) at $r = b$ and setting it equal to (6) gives C and hence the given interior potential. What we have carried out is an "inside-outside" calculation of the field distribution where the "inside" region is outside and the "outside" region is inside.

7.4.2 (a) This is an example of an inside-outside problem, where the potential is first determined in the conducting material. Because the current density normal to the outer surface is zero, this potential can be determined without regard for the geometry of what may be located outside. Then, given the potential on the surface, the outside potential is determined. Given the ϕ dependence of the normal current density at $r = b$, the potential in the conducting region is taken as having the form

$$\Phi^b = \left(Ar + \frac{B}{r^2}\right)\cos\theta \tag{1}$$

Boundary conditions are that

$$J_r = -\sigma\frac{\partial\Phi^b}{\partial r} = 0 \tag{2}$$

at $r = a$, which requires that $B = a^3A/2$ and that

$$-\sigma\frac{\partial\Phi^b}{\partial r} = J_o\cos\theta \tag{3}$$

at $r = b$. This condition together with the result of (2) gives $A = J_o/\sigma[(a/b)^3 - 1]$. Thus, the potential in the conductor is

$$\Phi^b = \frac{J_o a}{\sigma}\left[\frac{r}{a} + \frac{1}{2}(a/r)^2\right]\cos\theta/[(a/b)^3 - 1] \tag{4}$$

(b) The potential in the outside region must match that given by (4) at $r = a$. To match the θ dependence, a dipole potential is assumed and the coefficient adjusted to match (4) evaluated at $r = a$.

$$\Phi^a = \frac{3J_o a}{2\sigma[(a/b)^3 - 1]}(a/r)^2\cos\theta \tag{5}$$

7.4.3 (a) This is an inside-outside problem, where the region occupied by the conductor is determined without regard for what is above the interface except that at the interface the material above is insulating. The potential in the conductor must match the given potential in the plane $y = -a$ and must have no derivative with respect to y at $y = 0$. The latter condition is satisfied by using the cosh function for the y dependence and, in view of the x dependence of the potential at $y = -a$, taking the x dependence as also being $\cos(\beta x)$. The coefficient is adjusted so that the potential is then the given value at $y = 0$.

$$\Phi^b = V\frac{\cosh\beta y}{\cosh\beta a}\cos\beta x \tag{1}$$

(b) in the upper region, the potential must be that given by (1) in the plane $y = 0$ and must decay to zero as $y \to \infty$. Thus,

$$\Phi^a = V\frac{\cos\beta x}{\cosh\beta a}e^{-\beta y} \tag{2}$$

7.4.4 The potential is zero at $\phi = 0$ and $\phi = \pi/2$, so it is expanded in solutions to Laplace's equation that have multiple zeros in the ϕ direction. Because of the first of these conditions, these are solutions of the form

$$\Phi \propto r^{\pm n}\sin n\theta \tag{1}$$

To make the potential zero at $\phi = \pi/2$,

$$n\frac{\pi}{2} = \pi, 2\pi, \ldots \Rightarrow n = 2, 4, \ldots 2m; \quad m = 1, 2, 3, \ldots \tag{2}$$

Thus, the potential is assumed to take the form

$$\Phi = \sum_{m=1}^{\infty}(A_m r^{2m} + B_m r^{-2m})\sin 2m\phi \tag{3}$$

At the outer boundary there is no normal current density, so

$$\frac{\partial \Phi}{\partial r}(r = a) = 0 \tag{4}$$

and it follows from (3) that

$$2mA_m a^{2m-1} - 2mB_m a^{-2m-1} = 0 \Rightarrow B_m = A_m a^{4m} \tag{5}$$

At $r = b$, the potential takes the form

$$\Phi = \sum_{m=1}^{\infty} V_m \sin 2m\phi = v \tag{6}$$

The coefficients are evaluated as in (5.5.3) through (5.5.9).

$$V_n \frac{\pi}{4} \int_0^{\pi/2} v \sin 2n\theta d\theta = \frac{v}{n}; \quad n \text{ odd}$$

$$\Rightarrow V_m = \frac{4v}{\pi m} = A_m b^{2m} + B_m b^{-2m} = A_m (b^{2m} + a^{4m} b^{-2m}) \tag{7}$$

Thus,

$$A_m = 4v/m\pi b^{2m}[1 - (a/b)^{4m}] \tag{8}$$

Substitution of (8) and (5) into (3) results in the given potential.

7.4.5 (a) To make the ϕ derivative of the potential zero at $\phi = 0$ and $\phi = \alpha$, the ϕ dependence is made $\cos(n\pi\phi/\alpha)$. Thus, solutions to Laplace's equation in the conductor take the form

$$\Phi = \sum_{n=0}^{\infty} \left[A_n (r/b)^{(n\pi/\alpha)} + B_n (b/r)^{(n\pi/\alpha)}\right] \cos\left(\frac{n\pi\phi}{\alpha}\right) \tag{1}$$

where $n = 0, 1, 2, \ldots$ To make the radial derivative zero at $r = b$,

$$B_n = A_n \tag{2}$$

so that each term in the series

$$\Phi = \sum_{n=0}^{\infty} A_n \left[(r/b)^{(n\pi/\alpha)} + (b/r)^{(n\pi/\alpha)}\right] \cos\left(\frac{n\pi\phi}{\alpha}\right) \tag{3}$$

satisfies the boundary conditions on the first three of the four boundaries.

(b) The coefficients are now determined by requiring that the potential be that given on the boundary $r = a$. Evaluation of (3) at $r = a$, multiplication by $\cos(m\pi\phi/\alpha)$ and integration gives

$$
\begin{aligned}
-\frac{v}{2}\int_0^{\alpha/2} \cos\left(\frac{m\pi\phi}{\alpha}\right)d\phi + \frac{v}{2}\int_{\alpha/2}^{\alpha} \cos\left(\frac{m\pi}{\alpha}\phi\right)d\phi \\
&= \int_0^{\alpha} \sum_{n=0}^{\infty} A_n\left[(a/b)^{(n\pi/\alpha)} + (b/a)^{(n\pi/\alpha)}\right] \\
&\quad \cos\left(\frac{n\pi\phi}{\alpha}\right)\cos\left(\frac{m\pi\phi}{\alpha}\right)d\phi \\
&= -\frac{2\alpha}{m\pi}\sin\left(\frac{m\pi}{2}\right)
\end{aligned}
\tag{4}
$$

and it follows that (3) is the required potential with

$$
A_n = -\frac{2v}{n\pi}\sin\left(\frac{n\pi}{2}\right)/\left[(a/b)^{n\pi/\alpha} + (b/a)^{n\pi/\alpha}\right] \tag{5}
$$

7.4.6 To make the potential zero at $\phi = 0$ and $\phi = \pi/2$, the ϕ dependence is made $\sin(2n\phi)$. Then, the r dependence is divided into two parts, one arranged to be zero at $r = a$ and the other to be zero at $r = b$.

$$
\Phi = \sum_{n=1}^{\infty}\left\{A_n\left[(r/a)^{2n} - (a/r)^{2n}\right] + B_n\left[(r/b)^{2n} - (b/r)^{2n}\right]\right\}\sin(2n\phi) \tag{1}
$$

Thus, when this expression is evaluated on the outer and inner surfaces, the boundary conditions respectively involve only B_n and A_n.

$$
\Phi(r = a) = v_a = \sum_{n=1}^{\infty} B_n\left[(a/b)^{2n} - (b/a)^{2n}\right]\sin 2n\phi \tag{2}
$$

$$
\Phi(r = b) = v_b = \sum_{n=1}^{\infty} A_n\left[(b/a)^{2n} - (a/b)^{2n}\right]\sin 2n\phi \tag{3}
$$

To determine the B_n's, (2) is multiplied by $\sin(2m\phi)$ and integrated

$$
\int_0^{\pi/2} v_a \sin 2m\phi d\phi = \int_0^{\pi/2}\sum_{n=1}^{\infty} B_n\left[(a/b)^{2n} - (b/a)^{2n}\right]\sin 2n\phi \sin 2m\phi d\phi \tag{4}
$$

and it follows that for n even $B_n = 0$ while for n odd

$$
B_n = 4v_a/n\pi\left[(a/b)^{2n} - (b/a)^{2n}\right] \tag{5}
$$

A similar usage of (3) gives

$$A_n = 4v_b/n\pi[(b/a)^{2n} - (a/b)^{2n}] \tag{6}$$

By definition, the mutual conductance is the total current to the outer electrode when its voltage is zero divided by the applied voltage.

$$G = \frac{i_a|_{v_a=0}}{v_b} = -\frac{d\sigma}{v_b}\int_0^{\pi/2} \frac{\partial\Phi}{\partial r}\Big|_{r=a} a d\phi = -\frac{d\sigma a}{v_b}\int_0^{\pi/2}\sum_{n=1}^{\infty} A_n \frac{4n}{a}\sin 2n\phi d\phi \tag{7}$$

and it follows that the mutual conductance is

$$G = -\frac{d\sigma}{v_b}4\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{A_n}{n} = \frac{16 d\sigma}{\pi}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{1}{[(a/b)^{2n} - (b/a)^{2n}]n^2} \tag{8}$$

7.5 STEADY CURRENTS IN PIECE-WISE UNIFORM CONDUCTORS

7.5.1 To make the current density the given uniform value at infinity,

$$\Phi \rightarrow -\frac{J_o}{\sigma_a} r\cos\theta; \quad r \rightarrow \infty \tag{1}$$

At the surface of the sphere, where $r = R$

$$J_r^a = J_r^b \Rightarrow \sigma_a \frac{\partial\Phi^a}{\partial r} = \sigma_b \frac{\partial\Phi^b}{\partial r} \tag{2}$$

and

$$\Phi^a = \Phi^b \tag{3}$$

In view of the θ dependence of (1), select solutions of the form

$$\Phi^a = -\frac{J_o}{\sigma_a} r\cos\theta + A\frac{\cos\theta}{r^2}; \quad \Phi^b = Br\cos\theta \tag{4}$$

Substitution into (2) and (3) then gives

$$A = -\frac{J_o R^3}{\sigma_a}\frac{(\sigma_a - \sigma_b)}{(2\sigma_a + \sigma_b)}; \quad B = -\frac{J_o}{\sigma_a}\frac{3\sigma_a}{(2\sigma_a + \sigma_b)} \tag{5}$$

and hence the given solution.

7.5.2 These are examples of inside-outside approximations where the field in region (a) is determined first and is therefore the "inside" region.

(a) If $\sigma_b \gg \sigma_a$, then

$$\Phi^a(r = R) \approx \text{constant } = 0 \tag{1}$$

(b) The field must be $-(J_o/\sigma_a)\mathbf{i}_z$ far from the sphere and satisfy (1) at $r = R$. Thus, the field is the sum of the potential for the uniform field and a dipole field with the coefficient set to satisfy (1).

$$\Phi^a \approx -\frac{RJ_o}{\sigma_a}\Big[\frac{r}{R} - (R/r)^2\Big]\cos\theta \tag{2}$$

(c) At $r = R$, the normal current density is continuous and approximated by using (2). Thus, the radial current density at $r = R$ inside the sphere is

$$J_r^b(r = R) = J_r^a(r = R) = -\sigma_a \frac{\partial \Phi^a}{\partial r}\Big|_{r=R} = 2J_o\cos\theta \tag{3}$$

A solution to Laplace's equation having this dependence on θ is the potential of a uniform field, $\Phi = Br\cos(\theta)$. The coefficient B follows from (3) so that

$$\Phi^b \approx -\frac{3J_oR}{\sigma_b}(r/R)\cos\theta \tag{4}$$

In the limit where $\sigma_b \gg \sigma_a$, (2) and (4) agree with (a) of Prob. 7.5.1.

(d) In the opposite extreme, where $\sigma_a \gg \sigma_b$,

$$J_r^a(r = R) = 0 \tag{5}$$

Again, the potential is the sum of that due to the uniform field that prevails at infinity and a dipole solution. However, this time the coefficient is adjusted so that the radial derivative is zero at $r = R$.

$$\Phi^a \approx -\frac{RJ_o}{\sigma_a}\Big[\frac{r}{R} + \frac{1}{2}(R/r)^2\Big]\cos\theta \tag{6}$$

To determine the field inside the sphere, potential continuity is used. From (6), the potential at $r = R$ is $\Phi^b = -(3RJ_o/2\sigma_a)\cos\theta$ and it follows that inside the sphere

$$\Phi^b \approx -\frac{3}{2}\frac{RJ_o}{\sigma_a}(r/R)\cos\theta \tag{7}$$

In the limit where $\sigma_a \gg \sigma_b$, (a) of Prob. 7.5.1 agrees with (6) and (7).

7.5.3 (a) The given potential implies a uniform field, which is certainly irrotational and solenoidal. Further, it satisfies the potential conditions at $z = 0$ and $z = -l$ and implies that the current density normal to the top and bottom interfaces is zero. The given "inside" potential is therefore the correct solution.

(b) In the "outside" region above, boundary conditions are that

$$\Phi(x = 0, z) = -vz/l; \quad \Phi(x, 0) = 0;$$

$$\Phi(a, z) = 0; \quad \Phi(-l, x) = v(1 - \frac{x}{a}) \tag{1}$$

The potential must have the given linear dependence on the bottom horizontal interface and on the left vertical boundary. These conditions can be met by a solution to Laplace's equation of the form xz. By translating the origin of the x axis to be at $x = a$, the solution satisfying the boundary conditions on the top and right boundaries is of the form

$$\Phi = A(a - x)z = -\frac{v}{la}(a - x)z \tag{2}$$

where in view of (1a) and (1c), setting the coefficient $A = -v/l$ makes the potential satisfy conditions at the remaining two boundaries.

(c) In the air and in the uniformly conducting slab, the bulk charge density, ρ_u, must be zero. At its horizontal upper interface,

$$\sigma_u = \epsilon_a E_x^a - \epsilon_b E_x^b = -\epsilon_o vz/la \tag{3}$$

Note that $z < 0$ so if $v > 0$, $\sigma_u > 0$ as expected intuitively. The surface charge density on the lower surface of the conductor cannot be specified until the nature of the region below the plane $x = -b$ is specified.

(d) The boundary conditions on the lower "inside" region are homogeneous and do not depend on the "outside" region. Therefore the solution is the same as in (a). The potential in the upper "outside" region is one associated with a uniform electric field that is perpendicular to the upper electrode. To satisfy the condition that the tangential electric field be the same just above the interface as below, and hence the same at any location on the interface, this field must be uniform. If it is to be uniform throughout the air-space, it must be the same above the interface as in the region where the bounding conductors are parallel plates. Thus,

$$\mathbf{E} = \frac{v}{a}\mathbf{i}_x + \frac{v}{l}\mathbf{i}_z \tag{4}$$

The associated potential that is zero at $z = 0$ and indeed on the surface of the electrode where $x = -za/l$ is

$$\Phi = \frac{v}{a}x + \frac{v}{l}z \tag{5}$$

Finally, instead of (3), the surface charge density is now

$$\sigma_u = \epsilon_o v/a$$

7.5.4 (a) Because they are surrounded by either surfaces on which the potential is constrained or by insulating regions, the fields within the conductors are determined without regard for either the fields within the square or outside, where not enough information has been given to determine the fields. The condition that there be no normal current density, and hence no normal electric field intensity on the surfaces of the conductors that interface the insulating regions, is automatically met by having uniform fields in the conductors. Because these fields are normal to the electrodes that terminate these regions, the boundary conditions on these surfaces are met as well. Thus, regardless of what d is relative to a, in the upper conductor,

$$\mathbf{E} = -\mathbf{i}_x \frac{v}{a}; \quad \Phi = \frac{v}{a}x; \quad \mathbf{J} = -\sigma\frac{v}{a}\mathbf{i}_x \tag{1}$$

while in the conductor to the right

$$\mathbf{E} = -\mathbf{i}_y \frac{v}{a}; \quad \Phi = \frac{v}{a}y; \quad \mathbf{J} = -\sigma\frac{v}{a}\mathbf{i}_y \tag{2}$$

(b) In the planes $y = a$ and $x = a$ the potential inside must be the same as given by (1) and (2) in these planes, linear functions of x and of y, respectively. It must also be zero in the planes $x = 0$ and $y = 0$. A simple solution meeting these conditions is

$$\Phi = Axy = \frac{v}{a^2}xy \tag{3}$$

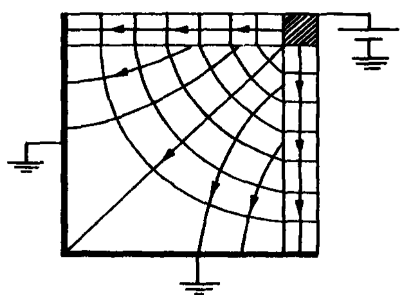


Figure S7.5.4

(c) The distribution of potential and electric field intensity is as shown in Fig. S7.5.4.

7.5.5 (a) Because the potential difference between the plates, either to the left or to the right, is zero, the electric field there must be zero and the potential that of the respective electrodes.

$$\Phi^a \to 0 \text{ as } x \to \infty; \quad \Phi^b \to v \text{ as } x \to -\infty \tag{1}$$

(b) Solutions that satisfy the boundary conditions on all but the interface at $x = 0$ are

$$\Phi^a = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi}{a}x} \sin\frac{n\pi}{a}y \tag{2a}$$

$$\Phi^b = v + \sum_{n=1}^{\infty} B_n e^{n\pi x/a} \sin \frac{n\pi}{a} y \tag{2b}$$

(c) At the interface, boundary conditions are

$$-\sigma_a \frac{\partial \Phi^a}{\partial x} = -\sigma_b \frac{\partial \Phi^b}{\partial x} \tag{3}$$

$$\Phi^a = \Phi^b \tag{4}$$

(d) The first of these requires of (2) that

$$\sigma_a \frac{n\pi}{a} A_n = -\sigma_b \frac{n\pi}{a} B_n \Rightarrow B_n = -\frac{\sigma_a}{\sigma_b} A_n \tag{5}$$

Written using this, the second requires that

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{a} y = v - \sum_{n=1}^{\infty} \frac{\sigma_a}{\sigma_b} A_n \sin \frac{n\pi}{a} y \tag{6}$$

The constant term can also be written as a Fourier series using an evaluation of the coefficients that is essentially the same as in (5.5.3)-(5.5.9).

$$v = \sum_{n=1}^{\infty} \frac{4v}{\pi n} \sin \frac{n\pi}{a} y \tag{7}$$

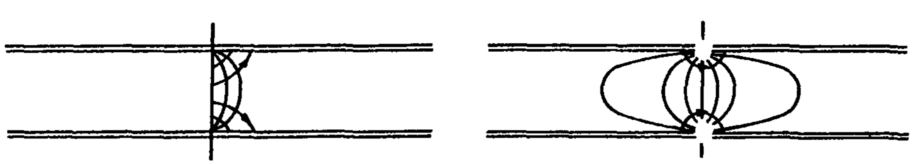
Thus,

$$A_n \left(1 + \frac{\sigma_a}{\sigma_b}\right) = \frac{4v}{n\pi} \tag{8}$$

and it follows that the required potential is

$$\Phi^a = \sum_{n=1}^{\infty} \frac{4v}{n\pi(1 + \sigma_a/\sigma_b)} e^{-\frac{n\pi}{a}x} \sin \frac{n\pi}{a} y$$

$$\Phi^b = v - \sum_{n=1}^{\infty} \frac{\sigma_a}{\sigma_b} \frac{4v e^{\frac{n\pi}{a}x}}{n\pi(1 + \sigma_a/\sigma_b)} \sin \frac{n\pi}{a} y \tag{9}$$



$\sigma_b \gg \sigma_a$ $\sigma_a = \sigma_b$

Figure S7.5.5

(e) In the case where $\sigma_b \gg \sigma_a$, the "inside" region is to the left where boundary conditions are on the potential at the upper and lower surfaces and on its normal derivative at the interface. In this limit, the potential is uniform throughout the region and the interface is an equipotential having $\Phi = v$. Thus, the potential in the region to the right is as shown in Fig. 5.5.3 with the surface at $y = b$ playing the role of the interface and the surface at $y = 0$ at infinity. In the case where the region between electrodes is filled by a uniform conductor, the potential and field distribution are as sketched in Fig. S7.5.5. In the vicinity of the regions where the electrodes abut, the potential becomes that illustrated in Fig. 5.7.2. By symmetry, the plane $x = 0$ is one having the potential $\Phi = v/2$.

(f) The surface at $y = a/2$ is a plane of symmetry in the previous configuration and hence one where $E_y = 0$. Thus, the previous solution applies directly to finding the solution in the conducting layer.

7.6 CONDUCTION ANALOGS

7.6.1 The analogous laws are

$$\mathbf{E} = -\nabla\Phi \qquad \mathbf{E} = -\nabla\Phi \tag{1}$$

$$\nabla \cdot \sigma\mathbf{E} = s \qquad \nabla \cdot \epsilon\mathbf{E} = \rho_u \tag{2}$$

The systems are normalized to different length scales. The conductivity and permittivity are respectively normalized to σ_c and ϵ_ϵ respectively and similarly, the potentials are normalized to the respective voltages V_c and V_ϵ.

$$(x, y, z) = (\underline{x}, \underline{y}, \underline{z})l_c \qquad (x, y, z) = (\underline{x}, \underline{y}, \underline{z})l_\epsilon \tag{3}$$

$$\Phi = V_c\underline{\Phi} \qquad \Phi = V_\epsilon\underline{\Phi} \tag{4}$$

$$\mathbf{E} = (V_c/l_c)\underline{\Phi} \qquad \mathbf{E} = (V_\epsilon/l_\epsilon)\underline{\Phi} \tag{5}$$

$$s = (\sigma_c V_c/l_c^2)\underline{s} \tag{6}$$

$$\rho_u = (\epsilon_\epsilon V_\epsilon/l_\epsilon^2)\underline{\rho}_u \tag{7}$$

By definition, the normalized quantities are the same in the two systems

$$\underline{\sigma}(\mathbf{r}) = \underline{\epsilon}(r) \tag{8}$$

$$\underline{s}(r) = \underline{\rho}_u(r) \tag{9}$$

so that both systems are represented by the same normalized laws.

$$\underline{\mathbf{E}} = -\underline{\nabla}\underline{\Phi} \tag{10}$$

$$\underline{\nabla}\cdot\underline{\sigma\mathbf{E}} = \underline{s} \tag{11}$$

Thus, the capacitance and conductance are respectively

$$C = \epsilon_\epsilon l_\epsilon \oint_S \underline{\epsilon\mathbf{E}}\cdot d\underline{a} \Big/ \int_C \underline{\mathbf{E}}\cdot d\underline{s} \tag{12}$$

$$G = \sigma_c l_c \oint_S \underline{\sigma\mathbf{E}}\cdot d\underline{a} \Big/ \int_C \underline{\mathbf{E}}\cdot d\underline{a} \tag{13}$$

where, again by definition, the normalized integral ratios in (12) and (13) are the same number. Thus,

$$C/G = \frac{\epsilon_\epsilon}{\sigma_c}\frac{l_\epsilon}{l_c} = \frac{\epsilon}{\sigma}\frac{l_\epsilon}{l_c} \tag{14}$$

Note that the deductions summarized by (7.6.3) could be made following the same normalization approach.

7.7 CHARGE RELAXATION IN UNIFORM CONDUCTORS

7.7.1 (a) The charge is given when $t = 0$

$$\rho = \rho_i \sin\frac{\pi}{a}x \sin\frac{\pi}{b}y \tag{1}$$

Given the charge density, none of the bulk or surface conditions needed to determine the field involve time rates of change. Thus, the initial potential distribution is determined from the initial conditions alone.

(b) The properties of the region are uniform, so (3) and hence (4) apply directly. Given the charge is (c) of Prob. 4.1.4 when $t = 0$, the subsequent distribution of charge is

$$\rho = \rho_o(t)\sin\frac{\pi}{a}x\sin\frac{\pi}{b}y; \quad \rho_o = \rho_i e^{-t/\tau}; \quad \tau \equiv \frac{\epsilon}{\sigma} \tag{2}$$

(c) As in (a), at each instant the charge density is known and all other conditions are independent of time rates of change. Thus, the potential and field distributions simply go along with the changing charge density. They follow from (a) and (b) of Prob. 4.1.4 with $\rho_o(t)$ given by (2).

(d) Again, with $\rho_o(t)$ given by (2), the current is given by (6) of Prob. 4.1.4.

7.7.2 (a) The line charge is pictured as existing in the same uniformly conducting material as occupies the surrounding region. Thus, (7.7.3) provides the solution.

$$\lambda_l = \lambda_l(t = 0)e^{-t/\tau}; \quad \tau = \epsilon/\sigma \tag{1}$$

(b) There is no initial charge density in the surrounding region. Thus, the charge density there is zero.

(c) The potential is given by (1) of Prob. 4.5.4 with λ_l given by (1).

7.7.3 (a) With $q < -q_c$, the entire surface of the particle can collect the ions. Equation (7.7.10) becomes simply

$$i = -\mu\rho 6\pi R^2 E_a \int_0^{\pi} \Big(\cos\theta + \frac{q}{q_c}\Big) \sin\theta d\theta \tag{1}$$

Integration and the definition of q_c results in the given current.

(b) The current found in (a) is equal to the rate at which the charge on the particle is increasing.

$$\frac{dq}{dt} = -\frac{\mu\rho}{\epsilon} q \tag{2}$$

This expression can either be formally integrated or recognized to have an exponential solution. In either case, with $q(t=0) = q_o$,

$$q = q_o e^{-t/\tau}; \quad \tau \equiv \epsilon/\mu\rho \tag{3}$$

7.7.4 The potential is given by (5.9.13) with q replaced by q_c as defined with (7.7.11)

$$\Phi = -E_a R \cos\theta \Big[\frac{r}{R} - (R/r)^2\Big] + \frac{12\pi\epsilon_o R^2 E_a}{4\pi\epsilon_o r} \tag{1}$$

The reference potential as $r \to \infty$ with $\theta = \pi/2$ is zero. Evaluation of (1) at $r = R$ therefore gives the particle potential relative to infinity in the plane $\theta = \pi/2$.

$$\Phi = 3RE_a \tag{2}$$

The particle charges until it reaches 3 times a potential equal to the radius of the particle multiplied by the ambient field.

7.8 ELECTROQUASISTATIC CONDUCTION LAWS FOR INHOMOGENEOUS MATERIAL

7.8.1 For $t < 0$, steady conduction prevails, so $\partial(\ )/\partial t = 0$ and the field distribution is defined by (7.4.1)

$$\nabla \cdot (\sigma \nabla \Phi) = -s \tag{1}$$

where

$$\Phi = \Phi_\Sigma \quad \text{on} \quad S'; \qquad -\sigma\nabla\Phi = \mathbf{J}_\Sigma \quad \text{on} \quad S'' \tag{2}$$

To see that the solution to (1) subject to the boundary conditions of (2) is unique, propose different solutions Φ_a and Φ_b and define the difference between these solutions as

$$\Phi_d = \Phi_a - \Phi_b \tag{3}$$

Then it follows from (1) and (2) that

$$\nabla \cdot (\sigma \nabla \Phi_d) = 0 \tag{4}$$

where

$$\Phi_d = 0 \quad \text{on} \quad S'; \qquad -\sigma \nabla \Phi_d = 0 \quad \text{on} \quad S'' \tag{5}$$

Multiplication of (4) by Φ_d and integration over the volume V of interest gives

$$\int_V \Phi_d \nabla \cdot (\sigma \nabla \Phi_d) dv = 0 = \int_V [\nabla \cdot (\Phi_d \sigma \nabla \Phi_d) - \sigma \nabla \Phi_d \cdot \nabla \Phi_d] dv \tag{6}$$

Gauss' theorem converts this expression to

$$\oint_S \Phi_d \sigma \nabla \Phi_d \cdot d\mathbf{a} = \int_V \sigma \nabla \Phi_d \cdot \nabla \Phi_d dv \tag{7}$$

The surface integral can be broken into one on S', where $\Phi_d = 0$ and one on S'', where $\sigma \nabla \Phi_d = 0$. Thus, what is on the left in (7) is zero. If the integrand of what is on the right were finite anywhere, the integral could not be zero, so we conclude that to within a constant, $\Phi_d = 0$ and the steady solution is unique.

For $0 < t$, the steps beginning with (7.8.11) and leading to (7.8.15) apply. Again, the surface integration of (7.8.11) can be broken into two parts, one on S' where $\Phi_d = 0$ and one on S'' where $-\sigma \nabla \Phi_d = 0$. Thus, (7.8.16) and its implications for the uniqueness of the solution apply here as well.

7.9 CHARGE RELAXATION IN UNIFORM AND PIECE-WISE UNIFORM SYSTEMS

7.9.1 (a) In the first configuration, the electric field is postulated to be uniform throughout the gap and therefore the same as though the lossy segment were not present.

$$\mathbf{E} = \mathbf{i}_r v / r ln(a/b) \tag{1}$$

This field is irrotational and solenoidal and integrates to v between $r = b$ and $r = a$. Note that the boundary conditions at the interfaces between the lossy-dielectric and the free space region are automatically met. The tangential electric field (and hence the potential) is indeed continuous and, because there is no normal component of the electric field at these interfaces, (7.9.12) is satisfied as well.

(b) In the second configuration, the field is assumed to take the piece wise form

$$\mathbf{E} = \mathbf{i}_r \text{Re} \frac{1}{r} \begin{Bmatrix} \hat{A} \\ \hat{B} \end{Bmatrix} e^{j\omega t} \qquad \begin{matrix} R < r < a \\ b < r < R \end{matrix} \tag{2}$$

where A and B are determined by the requirements that the applied voltage be consistent with the integration of $\mathbf{E}$ between the electrodes and that (7.9.12) be satisfied at the interface.

$$\hat{A} ln(a/R) + \hat{B} ln(R/b) = \hat{v} \tag{3}$$

$$j\omega\left[\frac{\epsilon_o \hat{A}}{R} - \frac{\epsilon_b \hat{B}}{R}\right] - \frac{\sigma \hat{B}}{R} = 0 \tag{4}$$

It follows that

$$\hat{A} = (j\omega\epsilon_b + \sigma)\hat{v}/\mathrm{Det} \tag{5}$$

$$\hat{B} = j\omega\epsilon_o \hat{v}/\mathrm{Det} \tag{6}$$

where Det is as given and the relations that result from substitution of these coefficients into (2) are those given.

(c) In the first case, the net current to the inner electrode is

$$\hat{i} = j\omega l[(2\pi - \alpha)b\epsilon_o + \alpha b\epsilon]\frac{\hat{v}}{b ln(a/b)} + \frac{l\alpha b\sigma\hat{v}}{b ln(a/b)} \tag{7}$$

This expression takes the form of the impedance of a resistor in parallel with a capacitor where

$$\hat{i} = \hat{v}G + j\omega C\hat{v} \tag{8}$$

Thus, the C and G are as given in the problem.

In the second case, the equivalent circuit is given by Fig. 7.9.5 which implies that

$$\hat{i} = \frac{\hat{v}(j\omega C_a)(1 + j\omega RC_b)}{1 + j\omega R(C_a + C_b)} \tag{9}$$

In this case, the current to the inner electrode follows from (6) as

$$\hat{i} = \frac{2\pi l \frac{j\omega\epsilon_o}{ln(a/R)}\left(1 + \frac{j\omega\epsilon}{\sigma}\right)}{1 + \frac{j\omega ln(R/b)}{\sigma}\left[\frac{\epsilon_o}{ln(a/R)} + \frac{\epsilon}{ln(r/b)}\right]} \tag{10}$$

Comparison of these last two expressions results in the given parameters.

7.9.2 (a) <u>In the first case</u>, where the interface between materials is conical, the electric field intensity is what it would be in the absence of the material.

$$\mathbf{E} = \frac{v}{r^2}\frac{ab}{(a-b)}\mathbf{i}_r = \mathrm{Re}\,\frac{\hat{v}e^{j\omega t}}{r^2}\frac{ab}{(a-b)}\mathbf{i}_r; \quad \hat{v} \equiv V_o \tag{1}$$

This field is perpendicular to the perfectly conducting electrodes, has a continuous tangential component at the interface and trivially satisfies the condition of charge conservation at the interface.

In the second case, where the interface between materials is spherical, the field takes the form

$$\mathbf{E} = \mathbf{i}_r \mathrm{Re} \begin{cases} \hat{A}/r^2; & R < r < a \\ \hat{B}/r^2; & b < r < R \end{cases} \tag{2}$$

The coefficients are adjusted to satisfy the condition that the integral of $\mathbf{E}$ from $r = b$ to $r = a$ be equal to the voltage,

$$\hat{A}\Big(\frac{1}{R} - \frac{1}{a}\Big) + \hat{B}\Big(\frac{1}{b} - \frac{1}{R}\Big) = \hat{v} \tag{3}$$

and conservation of charge at the interface, (7.9.12).

$$-\frac{\sigma}{R^2}\hat{B} + j\omega\Big(\epsilon_o\frac{\hat{A}}{R^2} - \epsilon\frac{\hat{B}}{R^2}\Big) = 0 \tag{4}$$

Simultaneous solution of these expressions gives

$$\hat{A} = (\sigma + j\omega\epsilon)\hat{v}/\mathrm{Det} \tag{5}$$

$$\hat{B} = j\omega\epsilon_o/\mathrm{Det}$$

where

$$\mathrm{Det} \equiv \sigma\Big(\frac{a-R}{aR}\Big) + j\omega\Big[\epsilon\Big(\frac{a-R}{aR}\Big) + \epsilon_o\Big(\frac{R-b}{bR}\Big)\Big]$$

which together with (2) give the required field.

(b) In the first case, the inner electrode area subtended by the conical region occupied by the material is $2\pi b^2[1-\cos(\alpha/2)]$. With the voltage represented as $v = \mathrm{Re}\,\hat{v}\exp(j\omega t)$, the current from the inner spherical electrode, which has the potential v, is

$$\begin{aligned}\hat{i} &= 2\pi b^2[1-\cos(\alpha/2)](\sigma + j\omega\epsilon)\Big(\frac{ab}{a-b}\Big)\frac{\hat{v}}{b^2} \\ &+ 2\pi b^2\Big[1 - \cos\Big(\frac{2\pi-\alpha}{2}\Big)\Big]j\omega\epsilon_o\Big(\frac{ab}{a-b}\Big)\frac{\hat{v}}{b^2}\end{aligned} \tag{6}$$

Equation (6) takes the same form as for the terminal variables of the circuit shown in Fig. S7.9.2a. Thus,

$$\hat{i} = [G + j\omega(C_a + C_b)]\hat{v} \tag{7}$$

$$G = 2\pi[1-\cos(\alpha/2)]\frac{ab\sigma}{a-b}$$

$$C_a = 2\pi\Big[1 - \cos\Big(\frac{2\pi-\alpha}{2}\Big)\Big]\frac{ab\epsilon_o}{a-b} \tag{8}$$

$$C_b = 2\pi[1-\cos(\alpha/2)]\frac{ab\epsilon}{a-b}$$

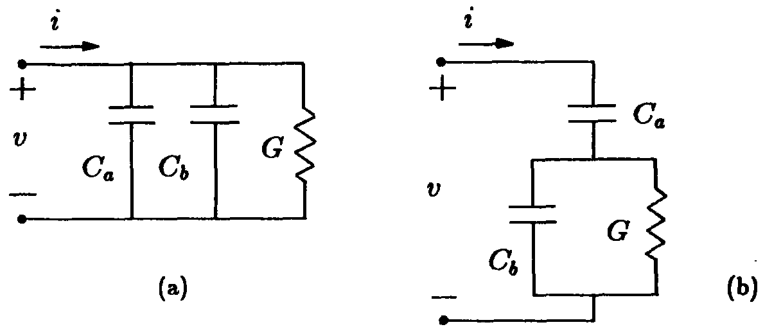


Figure S7.9.2

In the second case, the current from the inner electrode is

$$\hat{i} = 4\pi b^2\left(\frac{\sigma\hat{B}}{b^2} + j\omega\epsilon\frac{\hat{B}}{b^2}\right) \tag{9}$$

$$= \frac{j\omega\left(\frac{4\pi aR\epsilon_o}{a-R}\right)\left(\frac{4\pi Rb\sigma}{R-b} + j\omega\frac{4\pi Rb}{R-b}\epsilon\right)}{\frac{4\pi Rb\sigma}{R-b} - j\omega\left(\frac{4\pi Rb\epsilon}{R-b} + \frac{4\pi aR\epsilon_o}{a-R}\right)} \tag{10}$$

This takes the same form as the relationship between the terminal voltage and current for the circuit shown in Fig. S7.9.2b.

$$\hat{i} = \frac{j\omega C_a(G + j\omega C_b)}{G + j\omega(C_a + C_b)}\hat{v} \tag{11}$$

Thus, the elements in the equivalent circuit are

$$G = 4\pi\frac{Rb\sigma}{R-b};\quad C_a = \frac{4\pi Rb\epsilon_o}{R-b};\quad C_b = \frac{4\pi Rb\epsilon}{R-b} \tag{12}$$

7.9.3 In terms of the potential, v, of the electrode, the potential distribution and hence field distribution are

$$\Phi = v(a/r) \Rightarrow \mathbf{E} = \mathbf{i}_r\frac{va}{r^2} \tag{1}$$

The total current into the electrode is then equal to the sum of the rate of increase of the surface charge density on the interface between the electrode and the media and the conduction current from the electrode into the media.

$$i = \int_S \left[\frac{\partial}{\partial t}\epsilon E_r + \sigma E_r\right]da \tag{2}$$

In view of (1), this expression becomes

$$i = 2\pi a^2\left(\frac{\epsilon a}{a^2}\frac{dv}{dt} + \frac{\sigma a}{a^2}v\right) = 2\pi\epsilon a\frac{dv}{dt} + (2\pi\sigma a)v \tag{3}$$

The equivalent parameters are deduced by comparing this expression to one describing the current through a parallel capacitance and resistance.

√
7.9.4 (a) With A and B functions of time, the potential is assumed to have the same ϕ dependence as the applied field.

$$\Phi^a = -Er\cos\phi + A\frac{\cos\phi}{r} \tag{1}$$

$$\Phi^b = Br\cos\phi \tag{2}$$

The coefficients are determined by continuity of potential at $r = a$

$$\Phi^a(r = a) = \Phi^b(r = b) \tag{3}$$

and the combination of charge conservation and Gauss' continuity condition, also at $r = a$

$$(\sigma_a E_r^a - \sigma_b E_r^b) + \frac{\partial}{\partial t}(\epsilon_a E_r^a - \epsilon_b E_r^b) = 0 \tag{4}$$

Substitution of (1) and (2) into (3) and (4) gives

$$\frac{A}{a} - Ba = Ea \Rightarrow B = \frac{A}{a^2} - E \tag{5}$$

$$\sigma_a\left(E - \frac{A}{a^2}\right) + \sigma_b B + \frac{d}{dt}\left[\epsilon_a\left(E + \frac{A}{a^2}\right) + \epsilon_b B\right] = 0 \tag{6}$$

and from these relations,

$$(\epsilon_a + \epsilon_b)\frac{dA}{dt} + (\sigma_a + \sigma_b)A = (\sigma_b - \sigma_a)a^2 E + (\epsilon_b - \epsilon_a)a^2\frac{dE}{dt} \tag{7}$$

With E_o the magnitude of a step in $E(t)$, integration of (7) from $t = 0^-$ when $A = 0$ to $t = 0^+$ shows that

$$A(0^+) = \frac{\epsilon_b - \epsilon_a}{\epsilon_b + \epsilon_a}a^2 E_o \tag{8}$$

A particular solution to (7) for $t > 0$ is

$$A = \left(\frac{\sigma_b - \sigma_a}{\sigma_b + \sigma_a}\right)a^2 E_o \tag{9}$$

while a homogeneous solution is $\exp(-t/\tau)$, where

$$\tau \equiv \frac{\epsilon_a + \epsilon_b}{\sigma_a + \sigma_b} \tag{10}$$

Thus, the required solution takes the form

$$A = A_1 e^{-t/\tau} + \left(\frac{\sigma_b - \sigma_a}{\sigma_a + \sigma_b}\right)a^2 E_o$$

where the coefficient A_1 is determined by the initial condition, (8). Thus,

$$A = \left[\frac{(\epsilon_b - \epsilon_a)}{(\epsilon_b + \epsilon_a)} - \frac{(\sigma_b - \sigma_a)}{(\sigma_b + \sigma_a)}\right] a^2 E_o e^{-t/\tau} + \frac{(\sigma_b - \sigma_a)}{(\sigma_b + \sigma_a)} a^2 E_o \tag{12}$$

The coefficient B follows from (5).

$$B = \frac{A}{a^2} - E_o \tag{13}$$

In view of this last relation, and then (12), the unpaired surface charge density is

$$\begin{aligned} \sigma_{su} &= \epsilon_a\Big(E_o + \frac{A}{R^2}\Big) + \epsilon_b\Big(\frac{A}{R^2} - E_o\Big) \\ &= \frac{2(\epsilon_a\sigma_b - \epsilon_b\sigma_a)}{\sigma_a + \sigma_b} E_o(1 - e^{-t/\tau}) \end{aligned} \tag{14}$$

(b) In the sinusoidal steady state, the drive in (7) takes the form Re $\hat{E}\exp(j\omega t)$ and the resonse is of the form Re $\hat{A}\exp(j\omega t)$. Thus, (7) shows that

$$\hat{A} = \frac{[(\sigma_b - \sigma_a) + j\omega(\epsilon_b - \epsilon_a)]}{(\sigma_b + \sigma_a) + j\omega(\epsilon_b + \epsilon_a)} a^2 \hat{E}_p \tag{15}$$

and in turn, from (5),

$$\hat{B} = -\frac{2(\sigma_a + j\omega\epsilon_a)}{(\sigma_b + \sigma_a) + j\omega(\epsilon_b + \epsilon_a)} \tag{16}$$

This expressions can then be used to show that the complex amplitude of the unpaired surface charge density is

$$\hat{\sigma}_{su} = \frac{2(\sigma_b\epsilon_a - \sigma_a\epsilon_b)}{(\sigma_b + \sigma_a) + j\omega(\epsilon_b + \epsilon_a)} \tag{17}$$

(c) From (1) and (2) it is clear that the plane $\phi = \pi/2$ is one of zero potential, regardless of the values of the drive $E(t)$ or of A or B. Thus, the $x = 0$ plane can be replaced by a perfect conductor. In the limit where $\sigma_a \to 0$ and $\omega(\epsilon_a + \epsilon_b)/\sigma_b \ll 1$, (15) and (16) become

$$\hat{A} \to a^2 E_p \tag{18}$$

$$\hat{B} \to -\frac{2j\omega\epsilon_a}{\sigma_b}\hat{E}_p \tag{19}$$

Substitution of these coefficients into the sinusoidal steady state versions of (1) and (2) gives

$$\Phi^a = -\text{Re}\, \hat{E}_p a\Big[\frac{r}{a} - \frac{a}{r}\Big]\cos\phi e^{j\omega t} \tag{20}$$

$$\Phi^b = -\text{Re}\, \frac{2j\omega\epsilon_a}{\sigma_b}\hat{E}_p e^{j\omega t} \tag{21}$$

These are the potentials that would be obtained under sinusoidal steady state conditions using (a) and (b) of Prob. 7.9.5.

7.9.5 (a) This is an example of an "inside-outside" situation. The "inside" region is the one where the excitation is applied, namely region (a). In so far as the field in the exterior region is concerned, the surface is essentially an equipotential. Thus, the solution given by (a) must be constant at $r = a$ (it is zero), must become the uniform applied field at infinity (which it does) and must be comprised of solutions to Laplace's equation (which certainly the uniform and dipole fields are).

(b) To approximate the interior field, note that in general charge conservation and Gauss' law (7.9.12) require that

$$\frac{\partial}{\partial t}(\epsilon_o E_r^a - \epsilon E_r^b) - \sigma E_r^b = 0 \tag{1}$$

So long as the interior field is much less than that applied, this expression can be approximated by

$$\sigma E_r^b = \frac{\partial \epsilon_o E_r^a}{\partial t} = \epsilon_o 2\cos\phi \frac{dE}{dt} \tag{2}$$

which, in view of (a), is a prescription for the normal conduction current density inside the cylinder. This is then the boundary condition on the potential in region (b), the interior of the cylinder, and it follows that the potential within is

$$\Phi^b = Ar\cos\phi = -\frac{2\epsilon_o}{\sigma} r\cos\phi \frac{dE}{dt} \tag{3}$$

Note that the approximation made in going from (1) to (2) is valid if

$$\epsilon_o E_r^a \gg \epsilon E_r^b \Rightarrow \epsilon_o 2\cos\theta E \gg \frac{\epsilon}{\sigma}\epsilon_o 2\cos\phi \frac{dE}{dt} \tag{4}$$

Thus, if $E(t) = E_o \cos\omega t$, the approximation is valid provided

$$1 \gg \frac{\omega\epsilon}{\sigma} \tag{5}$$

√
7.9.6 (a) Just after the step, there has been no time for the relaxation of unpaired charge, so the system is still behaving as if the conductivity were zero. In any case, piece-wise solutions to Laplace's equation, having the same θ dependence as the dipole potential and having the dipole potential in the neighborhood of the origin are

$$\Phi^a = A\frac{\cos\theta}{r^2} \tag{1}$$

$$\Phi^b = \frac{p}{4\pi\epsilon_o}\frac{\cos\theta}{r^2} + Br\cos\theta \tag{2}$$

At $r = a$,

$$\Phi^a = \Phi^b \tag{3}$$

$$-\epsilon\frac{\partial\Phi^a}{\partial r} = -\epsilon_o\frac{\partial\Phi^b}{\partial r} \tag{4}$$

Substitution of (1) and (2) into these relations gives

$$\begin{bmatrix} \frac{1}{a^2} & -a \\ \frac{2\epsilon}{a^3} & \epsilon_o \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \frac{P}{4\pi\epsilon_o a^3}\begin{bmatrix} a \\ 1 \end{bmatrix} \tag{5}$$

Thus, the desired potentials are (1) and (2) evaluated using A and B found from (5) to be

$$A = \frac{3p}{4\pi(\epsilon_o + 2\epsilon)} \tag{6}$$

$$B = \frac{2(\epsilon_o - \epsilon)p}{4\pi\epsilon_o a^3(\epsilon_o + 2\epsilon)} \tag{7}$$

(b) After a long time, charge relaxes to the interface to render it an equipotential. Thus, the field outside is zero and that inside is determined by making B in (2) satisfy the condition that $\Phi^b(r = a) = 0$.

$$A = 0 \tag{8}$$

$$B = -\frac{p}{4\pi\epsilon_o a^3} \tag{9}$$

(c) In the general case, (4) is replaced by

$$\sigma\frac{\partial\Phi^a}{\partial r} + \frac{\partial}{\partial t}(\epsilon\frac{\partial\Phi^a}{\partial r} - \epsilon_o\frac{\partial\Phi^b}{\partial r}) = 0 \tag{10}$$

and substitution of (1) and (2) gives

$$2\sigma A + \frac{d}{dt}\left[2\epsilon A + \epsilon_o(\frac{-2p}{4\pi\epsilon_o} + Ba^3)\right] = 0 \tag{11}$$

With B replaced using (5a),

$$\frac{dA}{dt} + \frac{A}{\tau} = \frac{3}{4\pi(2\epsilon + \epsilon_o)}\frac{dp}{dt}; \quad \tau \equiv \frac{2\epsilon + \epsilon_o}{2\sigma} \tag{12}$$

With p a step function, integration of this expression from $t = 0^-$ to $t = 0^+$ gives

$$A(0^+) = \frac{2p_o}{4\pi(2\epsilon + \epsilon_o)} \tag{13}$$

It follows that

$$A = \frac{3p_o}{4\pi(2\epsilon + \epsilon_o)}e^{-t/\tau} \tag{14}$$

and in turn that

$$B = \frac{p_o}{4\pi a^3}\left(\frac{3e^{-t/\tau}}{2\epsilon + \epsilon_o} - \frac{1}{\epsilon_o}\right) \tag{15}$$

As $t \to 0$, these expressions become (6) and (7) while as $t \to \infty$, they are consistent with (8) and (9).

7.9.7 (a) This is an "inside-outside" situation where the layer of conductor is the "inside" region. The potential is constrained at the lower surface by the electrodes and the y derivative of the potential must be zero at the upper surface. This potential follows as

$$\Phi^b = V\frac{\cosh\beta y}{\cosh\beta d}\cos\beta x \tag{1}$$

The potential must be continuous at the upper interface, where it follows from (1) with $y = 0$ that it is

$$\Phi^a(y=0) = V\frac{\cos\beta x}{\cosh\beta d} \tag{2}$$

The potential that matches this condition in the plane $y = 0$ and goes to zero as y goes to infinity is

$$\Phi^a = V\frac{\cos\beta x}{\cosh\beta d}e^{-\beta y} \tag{3}$$

Thus, before $t = 0$, the surface charge density is

$$\sigma_{su} = -\left[\epsilon_o\frac{\partial\Phi^a}{\partial y} - \epsilon\frac{\partial\Phi^b}{\partial y}\right]_{y=0} = \frac{\epsilon_o\beta V\cos\beta x}{\cosh\beta d} \tag{4}$$

(b) Once the potential imposed by the lower electrodes is zero, the potentials in the respective regions take the form

$$\Phi^a = Ae^{-\beta y}\cos\beta x \tag{5a}$$

$$\Phi^b = A\frac{\sinh\beta(y+d)}{\sinh\beta d}\cos\beta x \tag{5b}$$

Here, the coefficients have been adjusted so that the potential is continuous at $y = 0$. The remaining condition to be satisfied at this interface is (7.9.12).

$$\frac{\partial}{\partial t}(\epsilon_o E_y^a - \epsilon E_y^b) - \sigma E_y^b = 0 \tag{6}$$

Substitution from (5) shows that

$$\frac{\partial}{\partial t}[(\epsilon_o\beta + \epsilon\coth\beta d)\cos\phi A] + \sigma\beta\coth\beta d\cos\phi A = 0 \tag{7}$$

The term inside the time derivative is the surface charge density. Thus, (7) can be converted to a differential equation for the surface charge density

$$\frac{d\sigma_{su}}{dt} + \frac{\sigma_{su}}{\tau} = 0 \tag{8}$$

where

$$\tau = (\epsilon_o\tanh\beta d + \epsilon)/\sigma$$

Thus, given the initial condition from part (a), the surface charge density is

$$\sigma_{su} = \frac{\epsilon_o\beta V\cos\beta x}{\cosh\beta d}e^{-t/\tau} \tag{9}$$

7.9.8 (a) Just after Q has been turned on, there is still no surface charge on the interface. Thus, when $t = 0^+$,

$$\Phi^a(y=0) = \Phi^b(y=0) \tag{1}$$

$$\epsilon_o \frac{\partial \Phi^a}{\partial y}(y=0) = \epsilon \frac{\partial \Phi^b}{\partial y}(y=0) \tag{2}$$

It follows from the postulated solutions that

$$Q - q_b = q_a \tag{3}$$

$$-\epsilon_o Q - \epsilon_o q_b = -\epsilon q_a \tag{4}$$

and finally that $q_a(0^+)$ and $q_b(0^+)$ have the given values.

(b) As $t \to \infty$, the interface becomes an equipotential. It follows from the postulated solution evaluated at the interface, where the potential must be what it is at infinity, namely zero, that

$$q_b = Q \tag{5}$$

(c) Throughout the transient, (1) must hold. However, the condition of (2) is generalized to represent the buildup of the surface charge density, (7.9.12). At $y = 0$

$$\frac{\partial}{\partial t}\left[-\epsilon_o \frac{\partial \Phi^a}{\partial y} + \epsilon \frac{\partial \Phi^b}{\partial y}\right] + \sigma \frac{\partial \Phi^b}{\partial y} = 0 \tag{6}$$

When $t > 0$, Q is a constant. Thus, evaluation of (6) with the postulated solutions gives

$$\epsilon_o \frac{dq_b}{dt} - \epsilon \frac{dq_a}{dt} - \sigma q_a = 0 \tag{7}$$

Using (1) to eliminate q_b, this expression becomes

$$\frac{dq_a}{dt} + \frac{q_a}{\tau} = 0 \tag{8}$$

where $\tau = (\epsilon_o + \epsilon)/\sigma$. The solution to this expression is $A\exp(-t/\tau)$, where A is the initial value found in part (a). The other image charge, q_b, is then given by using (1).

7.9.9 (a) As $t \to \infty$, the surface at $x = 0$ requires that there be no normal current density and hence electric field intensity on the (b) side. Thus, all boundary conditions in region (b) and Laplace's equation are satisfied in region (b) by a uniform electric field and a linear potential.

$$\mathbf{E}^b = \frac{v}{a}\mathbf{i}_y; \quad \Phi^b = \frac{v}{a}(a-y) \tag{1}$$

The field in region (a) can then be found. It has a potential that is zero on three of the four boundaries. On the fourth, where $x = 0$, the potential must be the same as given by (1)

$$\Phi^a(x=0) = \Phi^b(x=0) = \frac{v}{a}(a-y) \tag{2}$$

To match these boundary conditions, we take the solution to Laplace's equation to be an infinite sum of modes that satisfy the first three boundary conditions.

$$\Phi^a = -\sum_{n=1}^{\infty} A_n \frac{\sinh \frac{n\pi}{a}(x-b)}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi y}{a}\right) \tag{3}$$

The coefficients are determined by requiring that this sum satisfy the last boundary condition at $x = 0$.

$$\frac{v}{a}(a-y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{a}\right) \tag{4}$$

Multiplication by $\sin(m\pi y/a)$ and integration from $x = 0$ to $x = a$ gives

$$A_m = \frac{2}{a}\int_0^a \frac{v}{a}(a-y)\sin\left(\frac{m\pi y}{a}\right)dy = \frac{2v}{m\pi} \tag{5}$$

Thus, it follows that the potential in region (a) is

$$\Phi^a = -\sum_{n=1}^{\infty} \frac{2v}{n\pi} \frac{\sinh \frac{n\pi}{a}(x-b)}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi y}{a}\right) \tag{6}$$

(b) During the transient, the two regions are coupled by the temporal and spatial evoluation of unpaired charge at the interface, where $x = 0$. So, in region (b) we add to the asymptotic solution, which satisfies the conditions on the potential at $y = 0, y = a$ and as $x \to -\infty$, one that term-by-term is zero on these boundaries and as $x \to -\infty$ and that term-by-term satisfies Laplace's equation.

$$\Phi^b = \frac{v}{a}(a-y) + \sum_{n=1}^{\infty} B_n e^{n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \tag{7}$$

The result of (4)-(5) shows that the first term on the right can just as well be represented by the same Fourier series for its y dependence as the last term.

$$\Phi^b = \sum_{n=1}^{\infty} \frac{2v}{n\pi} \sin\left(\frac{n\pi y}{a}\right) + \sum_{n=1}^{\infty} B_n e^{n\pi x/a} \sin\left(\frac{n\pi y}{a}\right) \tag{8}$$

The potential in region (a) can generally take the form of (3). There remains finding $A_n(t)$ and $B_n(t)$ such that the continuity conditions at $x = 0$ on the potential and representing Gauss plus charge conservation are met. Evaluation of (3) and (8) at $x = 0$ shows that the potential continuity condition can be satisfied term-by-term if

$$A_n = \frac{2v}{n\pi} + B_n \tag{9}$$

The second condition brings in the dynamics, (7.9.12) at $x = 0$,

$$\left[\sigma_a \frac{\partial \Phi^a}{\partial x} - \sigma_b \frac{\partial \Phi^b}{\partial x}\right] + \frac{\partial}{\partial t}\left(\epsilon_a \frac{\partial \Phi^a}{\partial x} - \epsilon_b \frac{\partial \Phi^b}{\partial x}\right) = 0 \tag{10}$$

Substitution from (3) and (8) gives an expression that can also be satisfied term-by-term if

$$\begin{aligned} -\sigma_a \frac{n\pi}{a} \coth\left(\frac{n\pi b}{a}\right) A_n - \sigma_b \frac{n\pi}{a} B_n - \epsilon_a \frac{n\pi}{a} \coth\left(\frac{n\pi b}{a}\right) \frac{dA_n}{dt} \\ - \epsilon_b \frac{n\pi}{b} \frac{dB_n}{dt} = 0 \end{aligned} \tag{11}$$

Substitution for B_n from (9) then gives one expression that describes the temporal evolution of $A(t)$.

$$\frac{dA_n}{dt} + \frac{A_n}{\tau} = \frac{2}{n\pi}\left(\epsilon_b \frac{dv}{dt} + \sigma_b v\right) / \left(\epsilon_a \coth\left(\frac{n\pi b}{a}\right) + \epsilon_b\right) \tag{12}$$

where

$$\tau \equiv \frac{\epsilon_a \coth\left(\frac{n\pi b}{a}\right) + \epsilon_b}{\sigma_a \coth\left(\frac{n\pi b}{a}\right) + \sigma_b}$$

To find the response to a step, the volue of A_n when $t = 0$ is found by integrating (12) from $t = 0^-$ when $A_n = 0$ to $t = 0^+$.

$$A_n(0^+) = \epsilon_b V_o\left(\frac{2}{n\pi}\right) / \left(\epsilon_a \coth\left(\frac{n\pi b}{a}\right) + \epsilon_b\right) \tag{13}$$

The solution to (12), which takes the form of a homogeneous solution $\exp(-t/\tau)$ and a constant particular solution, must then satisfy this initial condition.

$$A_n = A_{n1} e^{-t/\tau} + \sigma_b\left(\frac{2}{n\pi}\right) \frac{V_o}{\left(\sigma_a \coth \frac{n\pi b}{a} + \sigma_b\right)} \tag{14}$$

The coefficient of the homogeneous term is adjusted to satisfy (13), and (14) becomes

$$\begin{aligned} A_n = V_o\left(\frac{2}{n\pi}\right)\Bigg\{ &\left[\frac{\sigma_b}{\sigma_a \coth\left(\frac{n\pi b}{a}\right) + \sigma_b} - \frac{\epsilon_b}{\epsilon_a \coth\left(\frac{n\pi b}{a}\right) + \epsilon_b}\right]\left(1 - e^{-t/\tau}\right) \\ &+ \frac{\epsilon_b}{\epsilon_a \coth\left(\frac{n\pi b}{a}\right) + \epsilon_b}\Bigg\} \end{aligned} \tag{15}$$

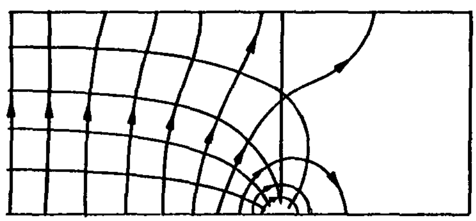
There is some insight gained by writing this expression in the alternative form

$$A_n = V_o\left(\frac{2}{n\pi}\right)\left[\frac{(\epsilon_a\sigma_b - \epsilon_b\sigma_a)\cosh\left(\frac{n\pi b}{2}\right)(1 - e^{-t/\tau})}{\sigma_a \coth\left(\frac{n\pi b}{a}\right) + \sigma_b} + \epsilon_b\right] \Big/ \left(\epsilon_a \coth\frac{n\pi b}{a} + \epsilon_b\right) \tag{16}$$

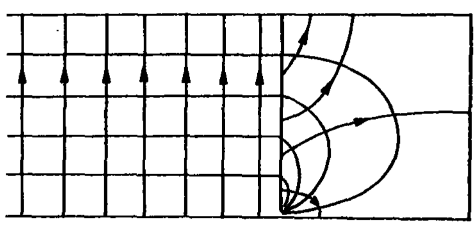
Given this expression for A_n, B_n follows from (9). In this specific situation these expressions are satisfied with $\sigma_a = 0$ and $\epsilon_a = \epsilon_o$.

(c) In this limit, it follows from (15) that as $t \to \infty$, $A_n \to V_o(2/n\pi)$ and this is consistent with what was found for this limit in part (a), (5).

With the permittivities equal, the potential and field distributions just after the potential has been turned on and therefore as there has been no time for unpaired charge to accumulate at the interface, is as shown in Fig. S7.9.9a. To make this sketch, note that far to the left, the equipotentials are equally spaced straight lines (surfaces) running parallel to the boundaries, which are themselves equipotentials. All of these must terminate in the gap at the origin. In the neighborhood of that gap, the potential has the form familiar from Fig. 5.7.2 (except that the equipotential $\Phi = V$ is at $\phi = \pi$ and not at $\phi = 2\pi$).



(a)



(b)

Figure S7.9.9

In the limit where $t \to \infty$, the uniform equipotentials in region (b) extend up to the interface. Just as we could solve for the field in region (b) and then for that in region (a), we can also draw the fields in that order. In region (a), the potential is linear in y in the plane $x = 0$ and zero on the other two boundaries. Thus, the equipotentials that originate on the boundary at

$x = 0$ at equal distances, must terminate in the gap, where they converge like equally spaced spokes on the hub of a wheel.

The transient that we have described takes the field distribution from that of Fig. S7.9.9a, where there is a conduction current normal to the interface from the (b) region side supplying surface charge to the interface, to that of Fig. S7.9.9b, where the current density normal to the surface has subsided because charges on the interface have created just that field necessary to null the normal field in region (b).