Also available from Prentice-Hall: Englewood Cliffs, NJ, 1990. ISBN: 9780132489805.

SOLUTIONS TO CHAPTER 1

1.1 THE LORENTZ LAW IN FREE SPACE

1.1.1 For $v_i = 0$, (7) gives

$$h = -\frac{1}{2}\frac{e}{m}E_x(t-t_i)^2 \Rightarrow t - t_i = \sqrt{\frac{-2hm}{\epsilon E_x}} \tag{1}$$

and from (8)

$$v = \sqrt{\frac{-2h\epsilon E_x}{m}} \tag{2}$$

so

$$v = \sqrt{\frac{2(1\times10^{-2})(1.602\times10^{-19})(10^{-2})}{(9.106\times10^{-31})}} = 5.9\times10^{31}\text{m/s} \tag{3}$$

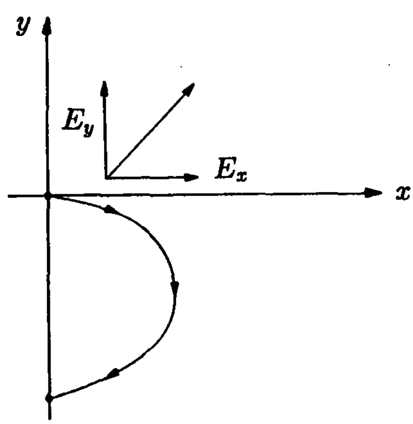

Figure S1.1.2

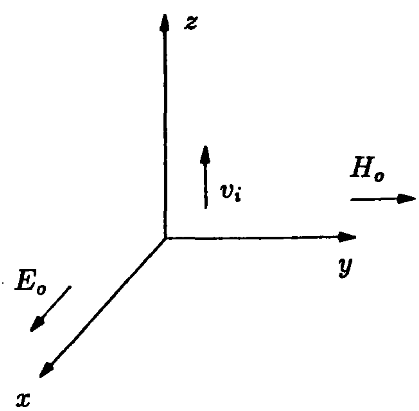

Figure S1.1.3

1.1.2 (a) In two-dimensions, (4) gives

$$\frac{md^2\xi_x}{dt^2} = -eE_x \tag{1}$$

$$\frac{md^2\xi_y}{dt^2} = -eE_y \tag{2}$$

so, because $v_x(0) = v_i$, while $v_y(0) = 0$,

$$\frac{d\xi_x}{dt} = -\frac{e}{m}E_x t + v_i \tag{3}$$

$$\frac{d\xi_y}{dt} = -\frac{e}{m}E_y t \tag{4}$$

To make $\xi_x(0) = 0$ and $\xi_y(0) = 0$

$$\xi_x = -\frac{e}{2m}E_x t^2 + v_i t \tag{5}$$

$$\xi_y = -\frac{e}{2m}E_y t^2 \tag{6}$$

(b) From (5), $\xi_x = 0$ when

$$t = \frac{v_i 2m}{eE_x} \tag{7}$$

and at this time

$$\xi_y = -\frac{e}{2m}E_y\left(\frac{v_i 2m}{eE_x}\right)^2 \tag{8}$$

1.1.3 The force is

$$\mathbf{f} = q[\mathbf{E} + \mathbf{v} \times \mu_o \mathbf{H}] = -e[E_o \mathbf{i_x} - v_z \mu_o H_o \mathbf{i_x}] \tag{1}$$

so, $\mathbf{f} = 0$ if $E_o = v_i \mu_o H_o$. Thus,

$$\frac{dv_x}{dt} = 0, \qquad \frac{dv_y}{dt} = 0, \qquad \frac{dv_z}{dt} = 0 \tag{2}$$

and v_x, v_y and v_z are constants. Because initial velocities in x and y directions are zero, $v_x = v_y = 0$ and $\mathbf{v} = v_i \mathbf{i_z}$.

1.1.4 The force is

$$\mathbf{f} = -e(\mathbf{E} + \mathbf{v} \times \mu_o \mathbf{H}) = -e(E_o \mathbf{i_y} + v_x \mu_o H_o \mathbf{i_z} - v_z \mu_o H_o \mathbf{i_x}) \tag{1}$$

so

$$\frac{md^2\xi_y}{dt^2} = -eE_o \Rightarrow \xi_y = -\frac{eE_o}{2m}t^2 \tag{2}$$

and

$$\frac{mdv_x}{dt} = ev_z \mu_o H_o \Rightarrow \frac{dv_x}{dt} = \omega_c v_z \tag{3}$$

$$\frac{mdv_z}{dt} = -ev_x \mu_o H_o \Rightarrow \frac{dv_z}{dt} = -\omega_c v_x \tag{4}$$

where $\omega_c = e\mu_o H_o/m$. Substitution of (3) into (4) gives

$$\frac{d^2v_x}{dt^2} + \omega_c^2 v_x = 0 \tag{5}$$

Solutions are $\sin \omega_c t$ and $\cos \omega_c t$. To satisfy the initial conditions on the velocity,

$$v_x = v_o \cos \omega_c t = \frac{d\xi_x}{dt} \tag{6}$$

in which case (3) gives:

$$v_z = -v_o \sin \omega_c t = \frac{d\xi_z}{dt} \tag{7}$$

Further integration and the initial conditions on $\bar{\xi}$ gives

$$\xi_x = \frac{v_o}{\omega_c} \sin \omega_c t \tag{8}$$

$$\xi_z = \frac{v_o}{\omega_c} \cos \omega_c t - \frac{v_o}{\omega_c} + z_o \tag{9}$$

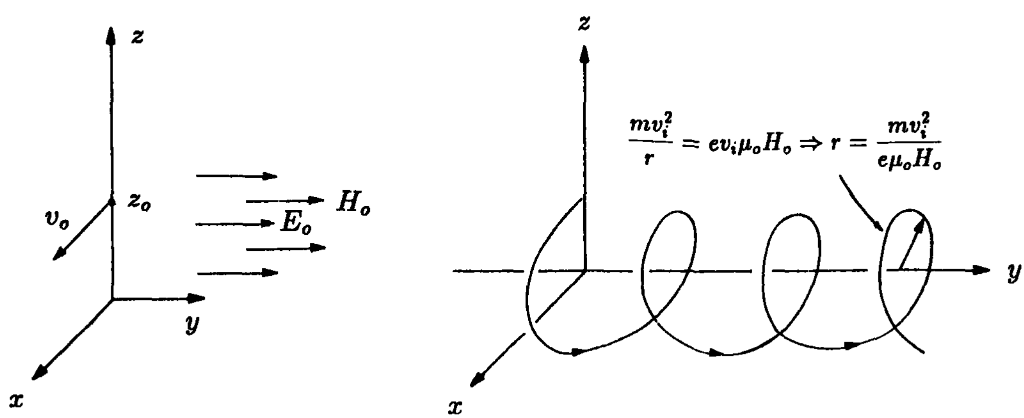

Figure S1.1.4

1.2 CHARGE AND CURRENT DENSITIES

1.2.1 The total charge is

$$q = \int_o^R \rho 4\pi r^2 dr = \int_o^R \frac{4\pi \rho_o r^3}{R} dr = \pi \rho_o R^3 \tag{1}$$

1.2.2 Integration of the density over the given volume gives the total charge

$$\begin{aligned} q &= \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a}\frac{\rho_o}{a^2}[x^2+y^2+z^2]dxdydz \\ &= \frac{2\rho_o}{a^2}\int_{-a}^{a}\int_{-a}^{a}(\frac{a^3}{3}+ay^2+az^2)dydz \end{aligned} \tag{1}$$

Two further integrations give

$$q = \frac{4\rho_o}{a^2}\int_{-a}^{a}(\frac{a^4}{3}+\frac{a^4}{3}+a^2z^2)dz = \frac{8\rho_o}{a^2}[\frac{2a^5}{3}+\frac{a^5}{3}] = 8\rho_o a^3 \tag{2}$$

1.2.3 The normal to the surface is $\mathbf{i_x}$, so

$$\begin{aligned} i &= \int_{-a}^{a}\int_{-a}^{a}\mathbf{J}\cdot\mathbf{n}dydz = \frac{J_o}{a^2}\int_{-a}^{a}\int_{-a}^{a}(y^2+z^2)dydz \\ &= \frac{2J_o}{a^2}\int_{-a}^{a}(\frac{1}{3}a^3+az^2)dz = \frac{8J_oa^2}{3} \end{aligned} \tag{1}$$

1.2.4 The net current is

$$i = \int_0^a J_o(\frac{r^2}{a^2})2\pi r dr = 2\pi\frac{J_o}{a^2}\frac{1}{4}r^4\Big|_0^a = \frac{\pi J_o a^2}{2} \tag{1}$$

1.2.5 (a) From Newton's second law

$$\frac{mdv_r}{dt} = -\frac{eE_ob}{\xi_r} \tag{1}$$

where

$$v_r = \frac{d\xi_r}{dt} \tag{2}$$

(b) On multiplying (1) by v_r,

$$mv_r\frac{dv_r}{dr} = -\frac{eE_ob}{\xi_r}v_r$$

and using (2), we obtain

$$mv_r\frac{dv_r}{dt} + \frac{eE_ob}{\xi_r}\frac{d\xi_r}{dt} = \frac{d}{dt}[\frac{1}{2}mv_r^2 + eE_obln\xi_r] = 0 \tag{3}$$

(c) Integrating (3) with respect to t gives

$$\frac{1}{2}mv_r^2 + eE_o b ln\xi_r = c_1 \tag{4}$$

When $t = 0, v_r = 0, \xi_r = b$ so $c_1 = eE_o b lnb$ and

$$\frac{1}{2}mv_r^2 + eE_o b ln\frac{\xi_r}{b} = 0 \tag{5}$$

Thus,

$$v_r(r) = \sqrt{\frac{2e}{m}E_o b ln\frac{b}{r}} \tag{6}$$

(d) The current density is

$$J_r = \rho(r)v_r(r) \Rightarrow \rho(r) = \frac{J_r}{v_r(r)} \tag{7}$$

The total current, i, must be independent of r, so

$$J_r = \frac{i}{2\pi r l} \tag{8}$$

and it follows from (6) and (7) that

$$\rho(r) = \frac{i}{2\pi r l}\sqrt{\frac{m}{2eE_o b ln(b/r)}} \tag{9}$$

1.3 GAUSS' INTEGRAL LAW OF ELECTRIC FIELD INTENSITY

1.3.1 (a) The unit vectors perpendicular to the 5 surfaces are as shown in Fig. S1.3.1. The given area elements follow from the same construction.

(b) From Fig. S1.3.1,

$$|\mathbf{i_r}|_x = \cos\phi = \frac{x}{\sqrt{x^2+y^2}}; \quad |\mathbf{i_r}|_y = \sin\phi = \frac{y}{\sqrt{x^2+y^2}} \tag{1}$$

$$r = \sqrt{x^2+y^2} \tag{2}$$

Thus, the conversion from polar to Cartesian coordinates gives

$$\mathbf{E} = \frac{\lambda_l}{2\pi\epsilon_o r}\mathbf{i_r} = \frac{\lambda_l}{2\pi\epsilon_o}\frac{1}{\sqrt{x^2+y^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\mathbf{i_x} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{i_y}\right) \tag{3}$$

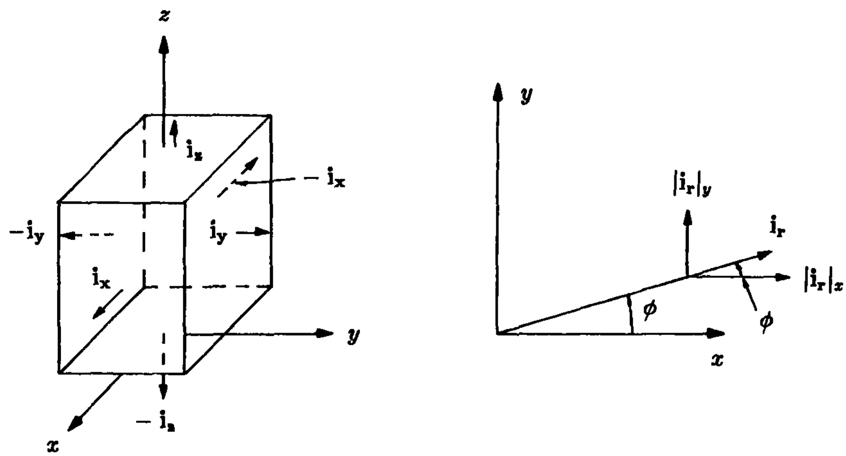

Figure S1.3.1

(c) On the given surface, the normal vector is $\mathbf{i}_x$ and so the integral is of the x component of (3) evaluated at $x = a$.

$$\int \epsilon_o \mathbf{E} \cdot d\mathbf{a}\Big|_{x=a} = \frac{\epsilon_o \lambda_l}{2\pi\epsilon_o} \int_0^1 \int_{-a}^{a} \frac{a}{a^2 + y^2} dy dz$$
$$= \frac{\lambda_l}{2\pi} \tan^{-1}\left(\frac{y}{a}\right)\Big|_{-a}^{a} = \frac{\lambda_l}{2\pi}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\lambda_l}{4} \tag{4}$$

Integration over the surface at $x = -a$ reverses both the sign of E_x and of the normal and so is also given by (4). Integrations over the surfaces at $y = a$ and $y = -a$ are respectively the same as given by (4), with the roles of x and y reversed. Integrations over the top and bottom surfaces make no contribution because there is no normal component of $\mathbf{E}$ on these surfaces. Thus, the total surface integration is four times that given by (4), which is indeed the charge enclosed, λ_l.

1.3.2 On the respective surfaces,

$$\mathbf{E} \cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_o} \begin{cases} 1/a^2 \\ 0 \\ 1/b^2 \end{cases} \tag{1}$$

On the two surfaces where these integrands are finite, they are also constant, so integration amounts to multiplication by the respective areas.

$$\oint_S \epsilon_o \mathbf{E} \cdot d\mathbf{a} = \frac{q\epsilon_o}{4\pi\epsilon_o}\left[\frac{1}{a^2}(2\pi a^2) + \frac{1}{b^2}(2\pi b^2)\right] = q \tag{2}$$

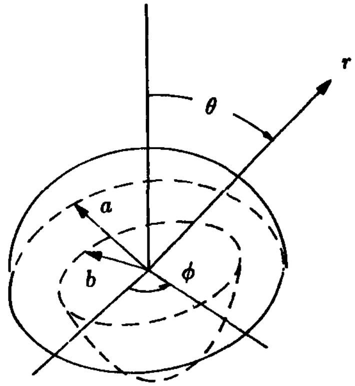

Figure S1.3.2

1.3.3 (a) Because of the axial symmetry, the electric field must be radial. Thus, integration of E_r over the surface at $r = r$ amounts to a multiplication by the area. For $r < b$, Gauss' integral law therefore gives

$$2\pi r l \epsilon_o E_r = \int_0^l \int_0^{2\pi} \int_0^r \rho dr r d\phi dz = 2\pi l \int_0^r \frac{\rho_o r^3}{b^2} dr \qquad (1)$$

$$E_r = \frac{\rho_o r^3}{4\epsilon_o b^2}; \quad r < b$$

For $b < r < a$, the integral on the right stops at $r = b$.

$$E_r = \frac{\rho_o b^2}{4\epsilon_o b^2}; \quad b < r < a \qquad (2)$$

(b) From (17)

$$\sigma_s = \mathbf{i}_r \cdot (\epsilon_o \mathbf{E}^a - \epsilon_o \mathbf{E}^b) = -\epsilon_o E_r(r = a) = -\frac{\rho_o b^2}{4a} \qquad (3)$$

(c) Because it is uniform there, integration of the surface charge density given by (3) over the surface $r = a$ amounts to a multiplication by the surface area.

$$\int \sigma_s da = \sigma_s 2\pi a l = -\rho_o b^2 \pi l/2 \qquad (4)$$

That this is the negative of the net charge within is confirmed by integrating over the enclosed charge density.

$$\int_V \rho dV = \int_0^l \int_0^{2\pi} \int_0^b \rho_o \left(\frac{r}{b}\right)^2 r d\phi dz = \frac{\pi l \rho_o b^2}{2} \qquad (5)$$

(d) As shown in the solution to Prob. 1.3.1,

$$\mathbf{i}_r = (x\mathbf{i}_x + y\mathbf{i}_y)/\sqrt{x^2+y^2}; \quad r = \sqrt{x^2+y^2} \tag{6}$$

and substitution into

$$\mathbf{E} = \frac{\rho_o}{4\epsilon_o}\begin{cases}(r^3/b^2)\mathbf{i}_r; & r < b \\ (b^2/r)\mathbf{i}_r; & b < r < a\end{cases} \tag{7}$$

indeed results in the given field distribution.

(e) For the surfaces at $x = \pm c$,

$$d\mathbf{a} = \pm \mathbf{i}_x dydz; \quad \mathbf{E}\cdot\mathbf{n} = E_x(x = \pm c) \tag{8}$$

while for those at $y = \pm c$,

$$d\mathbf{a} = \pm \mathbf{i}_y dxdz; \quad \mathbf{E}\cdot\mathbf{n} = E_y(y = \pm c) \tag{9}$$

The four terms in the given surface integral are the integrations over the respective surfaces using the field given by (d) evaluated in accordance with (8) and (9). According to (1), this integral must give the same answer as found by integrating the charge density over the enclosed volume. This has already been done and is given by (5).

1.3.4 (a) For $r < b$, (1) gives

$$4\pi r^2 \epsilon_o E_r = \int_0^r \rho_b 4\pi r^2 dr = \frac{4\pi\rho_b r^3}{3} \tag{1}$$

Thus,

$$E_r = \frac{\rho_o r}{3\epsilon_o}; \qquad r < b \tag{2}$$

Similarly, for $b < r < a$

$$4\pi r^2 \epsilon_o E_r = \frac{4}{3}\pi\rho_b b^3 + \int_b^r \frac{4\pi\rho_a r^2}{3} dr = \frac{4}{3}\pi[b^3\rho_b + (r^3 - b^3)\rho_a] \tag{3}$$

so that

$$E_r = \frac{1}{3\epsilon_o}\left[\frac{b^3\rho_b}{r^2} + \left(r - \frac{b^3}{r^2}\right)\rho_a\right]; \qquad b < r < a \tag{4}$$

(b) At $r = a$, (17) can be evaluated with $\mathbf{n} = \mathbf{i}_r, \mathbf{E}^a = 0$ and $\mathbf{E}^b$ given by (4)

$$\sigma_s = -\frac{1}{3}\left[\frac{b^3\rho_b}{a^2} + \left(a - \frac{b^3}{a^2}\right)\rho_a\right] \tag{5}$$

(c) For $r < b, E_r$ is still given by (2), while for $b < r < a$, (3) has an additional term on the right $4\pi b^2\sigma_o$. Thus,

$$E_r = \frac{1}{3\epsilon_o}\left[\frac{b^3\rho_b}{r^2} + \left(r - \frac{b^3}{r^2}\right)\rho_a\right] + \frac{b^2\sigma_o}{\epsilon_o r^2}; \qquad b < r < a \tag{6}$$

Then, instead of (5) we have

$$\sigma_s = -\frac{1}{3}\left[\frac{b^3\rho_b}{a^2} + \left(a - \frac{b^3}{a^2}\right)\rho_a\right] - \frac{b^2\sigma_o}{a^2} \tag{7}$$

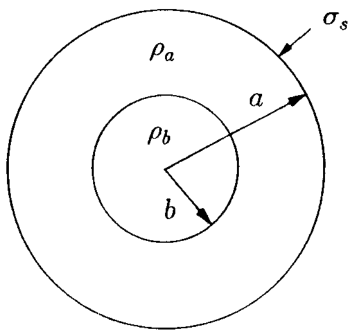

Figure S1.3.4

1.3.5 Using the volume described in Example 1.3.2, with the upper surface between the sheets, there is a contribution to the charge enclosed from both the lower sheet and the volume between that sheet and the position, z, of the upper surface. Thus, from (1)

$$\epsilon_o E_z(z) - \epsilon_o E_o = \sigma_o + \int_{-s/2}^{z} \rho dz = \sigma_o + \frac{2\rho_o z^2}{2s}\Big|_{-s/2}^{z} \quad (1)$$

and the solution for E_z gives

$$E_z = E_o + \frac{\sigma_o}{\epsilon_o} + \frac{\rho_o}{\epsilon_o s}\left[z^2 - (s/2)^2\right] \quad (2)$$

Note that the charge density is an odd function of z. Thus, there is no net charge between the sheets. With the surface above the upper sheet, the field given by (1) with the integration terminated at $z = s/2$ is just what it was below the lower sheet, E_o.

1.3.6 With the understanding that the charge distribution extends to infinity in the y and z directions, it follows from arguments already given that the electric field is independent of y and z and that that part of it due to the charge sheets can result only in a z directed electric field. It then follows from (1) that if the regions above and below the charge sustain no electric field intensity, then the net charge from the three layers must be zero. Thus, not only is

$$\sigma_a = 2\sigma_b \quad (1)$$

but also,

$$\sigma_a + \sigma_b + \sigma_o = 0 \quad (2)$$

From these relations, it follows that

$$\sigma_b = -\sigma_o/3; \qquad \sigma_a = -2\sigma_o/3 \quad (3)$$

1.3.7 The gravitational force has a component in the ξ direction, $-Mg\sin\alpha$. Thus, the sum of the forces acting on the upper particle in the ξ direction is

$$\frac{QQ_o}{4\pi\epsilon_o\xi^2} - Mg\sin\alpha = 0 \tag{1}$$

It follows that, for the particle to be in static equilibrium,

$$\xi = \sqrt{\frac{QQ_o}{4\pi\epsilon_o Mg\sin\alpha}} \tag{2}$$

1.4 AMPERE'S INTEGRAL LAW

1.4.1 Evaluation of (1) is carried out for a contour having the constant radius, r, on which symmetry requires that the magnetic field intensity be constant and in the ϕ direction. Because the fields are static, the last term on the right makes no contribution. Thus,

$$2\pi r H_\phi = \int_0^r J_r 2\pi r dr = \int_0^r 2\pi r J_o e^{-r/a} dr \tag{1}$$

Solving this expression for H_ϕ and carrying out the integration then gives

$$H_\phi = \frac{J_o}{r}\int_0^r re^{-r/a}dr = \frac{J_o}{r}a^2\big[1 - e^{-r/a}\big(1+\frac{r}{a}\big)\big] \tag{2}$$

1.4.2 (a) The net current carried by the wire in the $+z$ direction must be returned in the $-z$ direction on the surface at $r = a$. Thus,

$$\pi b^2 J_o + 2\pi a K_z = 0 \Rightarrow K_z = -\frac{b^2 J_o}{2a} \tag{1}$$

(b) For a contour at the constant radius, r, (1) is evaluated (with the last term on the right zero because the fields are static), first for $r < b$ and then for $b < r < a$.

$$2\pi r H_\phi = \int_0^r J_o 2\pi r dr = \pi r^2 J_o \Rightarrow H_\phi = \frac{J_o r}{2}; \qquad r < b \tag{2}$$

$$2\pi r H_\phi = \pi b^2 J_o \Rightarrow H_\phi = \frac{J_o b^2}{2r}; \qquad b < r < a \tag{3}$$

(c) From (1.4.16),

$$H_\phi^a - H_\phi^b = K_z \Rightarrow H_\phi^a = K_z + H_\phi^b \quad (4)$$

This expression can be evaluated using (1) and (3).

$$H_\phi^a = -\frac{b^2 J_o}{2a} + \frac{J_o b^2}{2a} = 0 \quad (5)$$

(d) In Cartesian coordinates,

$$H_x = -H_\phi \sin\phi = -H_\phi \frac{y}{\sqrt{x^2+y^2}}; \qquad H_y = H_\phi \cos\phi = H_\phi \frac{x}{\sqrt{x^2+y^2}} \quad (6)$$

Thus, with $r = \sqrt{x^2+y^2}$, evaluation of this expression using (2) and (3) gives

$$\mathbf{H} = \begin{cases} \frac{J_o}{2}\sqrt{x^2+y^2}(-y\mathbf{i_x} + x\mathbf{i_y})/\sqrt{x^2+y^2} = \frac{J_o}{2}(-y\mathbf{i_x} + x\mathbf{i_y}) \\ \frac{J_o b^2}{2\sqrt{x^2+y^2}} \frac{(-y\mathbf{i_x}+x\mathbf{i_y})}{\sqrt{x^2+y^2}} = \frac{J_o b^2}{2(x^2+y^2)}(-y\mathbf{i_x} + x\mathbf{i_y}) \end{cases} \quad (7)$$

(e) On $x = \pm c, \mathbf{H}\cdot d\mathbf{s} = \pm\mathbf{H}\cdot\mathbf{i_y}$ while on $y = \pm c, \mathbf{H}\cdot d\mathbf{s} = \mp\mathbf{H}\cdot\mathbf{i_x}$ so evaluation of (1) on the square contour gives

$$\begin{aligned} &\int_{-c}^{c}[H_x(x,-c) - H_x(x,c)]dx + \int_{-c}^{c}[H_y(c,y) - H_y(-c,y)]dy \\ &= \int_{-c}^{c}\frac{J_o b^2}{2}\left[\frac{c}{x^2+c^2} - \frac{(-c)}{x^2+y^2}\right]dx \\ &+ \int_{-c}^{c}\frac{J_o b^2}{2}\left[\frac{c}{c^2+y^2} - \frac{(-c)}{c^2+y^2}\right]dy \end{aligned} \quad (8)$$

The result of carrying out this integration must be equal to what is obtained by carrying out the surface integral on the right in (1).

$$\oint_C \mathbf{H}\cdot d\mathbf{s} = \int_S \mathbf{J}\cdot d\mathbf{a} = \pi b^2 J_o \quad (9)$$

1.4.3 (a) The total current in the $+z$ direction through the shell between $r = a$ and $r = b$ must equal that in the $-z$ direction through the wire at the center. Because the current density is uniform, it is then simply the total current divided by the cross-sectional area of the shell.

$$I = J_z[\pi(a^2 - b^2)] \Rightarrow J_z = I/\pi(a^2 - b^2) \quad (1)$$

(b) Ampère's integral law is written for a contour that circulates around the z axis at the constant radius r. The fields are constant, so the last term in (1.4.1) is zero. Symmetry arguments can be used to argue that $\mathbf{H}$ is ϕ directed and uniform on this contour, thus

$$2\pi r H_\phi = -I \Rightarrow H_\phi = -I/2\pi r; \qquad 0 < r < b \tag{2}$$

$$2\pi r H_\phi = -I + \frac{I}{\pi(a^2 - b^2)}\pi(r^2 - b^2) \Rightarrow H_\phi = I\left[-\frac{1}{2\pi r} + \frac{(r^2 - b^2)}{a^2 - b^2}\frac{1}{2\pi r}\right] \tag{3}$$

(c) Analysis of the ϕ directed H-field into Cartesian coordinates gives

$$\begin{aligned} H_x &= -H_\phi \sin\phi = -H_\phi y/\sqrt{x^2 + y^2} \\ H_y &= -H_\phi \cos\phi = H_\phi x/\sqrt{x^2 + y^2} \end{aligned} \tag{4}$$

where $r = \sqrt{x^2 + y^2}$. Thus, from (2) and (3),

$$\mathbf{H} = \frac{I(y\mathbf{i}_x - x\mathbf{i}_y)}{2\pi(x^2 + y^2)} \begin{cases} 1; & 0 < \sqrt{x^2 + y^2} < b \\ 1 - \frac{(x^2 + y^2 - b^2)}{a^2 - b^2}; & b < r < a \end{cases} \tag{5}$$

(d) In evaluating the line integral on the four segments of the square contour, on $x = \pm c$, $d\mathbf{s} = \pm\mathbf{i}_y dy$ and $\mathbf{H} \cdot d\mathbf{s} = \pm H_y(\pm c, y)dy$ while on $y = \pm c$, $d\mathbf{s} = \mp\mathbf{i}_x dx$ and $\mathbf{H} \cdot d\mathbf{s} = \mp H_x(x, \mp c)dx$. Thus,

$$\begin{aligned} \oint_C \mathbf{H} \cdot d\mathbf{s} = &\int_{-c}^{c} H_y(c, y)dy + \int_{-c}^{c} -H_x(x, -c)dx \\ &+ \int_{-c}^{c} -H_y(-c, y)dy + \int_{-c}^{c} H_x(x, c)dx \end{aligned} \tag{6}$$

This integral must be equal to the right hand side of (1.4.1), which can be evaluated in accordance with whether the contour stays within the region $r < b$ or is closed within the shell. In the latter case, the integration over the area of the shell enclosed by the contour is accomplished by simply multiplying the current density by the area of the square minus that of region inside the radius $r = b$.

$$\oint_S \mathbf{J} \cdot d\mathbf{a} =$$

$$\begin{cases} -I; & c < b/\sqrt{2} \\ -I + \frac{I}{\pi(a^2 - b^2)}[(2c)^2 - \pi b^2 + 4(\alpha b^2 - c\sqrt{b^2 - c^2})]; & b/\sqrt{2} < c < b \\ -I + \frac{I}{\pi(a^2 - b^2)}[(2c)^2 - \pi b^2]; & b < c < a/\sqrt{2} \end{cases} \tag{7}$$

where $\alpha = \cos^{-1}(c/b)$. The range $b/\sqrt{2} < c < b$ is complicated by the fact that the square contour overlaps the circle $r = b$. Thus, the area over which the return current in the shell passes through the square contour is the area of the square $(2c)^2$, minus the area of the region inside the radius b (as in the last case where there is no overlap of the square contour and the surface at $r = b$) plus the area where the circle $r = b$ extends beyond the square, which should not have been subtracted away.

1.4.4 (a) The net current passing through any plane of constant z must be zero. Thus,

$$2\pi a K_{za} + 2\pi b K_{zb} = I \tag{1}$$

and we are given that

$$K_{za} = 2K_{zb} \tag{2}$$

Solution of these expressions gives the desired surface current densities

$$K_{za} = \frac{I}{\pi(2a+b)}; \qquad K_{zb} = \frac{I}{2\pi(2a+b)} \tag{3}$$

(b) For $r < b$, Ampère's integral law, (1.4.1), applied to the region $r < b$ where the only current enclosed by the contour is due to that on the z axis, gives

$$2\pi r H_\phi = -I \Rightarrow H_\phi = \frac{-I}{2\pi r}; \qquad r < b \tag{4}$$

In the region $b < r < a$, the contour encloses the inner of the two surface current densities as well. Because it is in the z direction, its contribution is of opposite sign to that of I.

$$2\pi r H_\phi = -I + 2\pi b K_{zb} = -\left(\frac{2a}{2a+b}\right)I \tag{5}$$

Thus,

$$H_\phi = -\frac{I}{2\pi r}\left(\frac{2a}{2a+b}\right); \qquad b < r < a \tag{6}$$

Note that if Ampère's law is applied where $a < r$, the net current enclosed is zero and hence the magnetic field intensity is zero.

1.4.5 Symmetry arguments can be used to show that $\mathbf{H}$ depends only on z. Ampère's integral law is used with a contour that is in a plane of constant y, so that it encloses the given surface and volume currents. With z taken to be in the vertical direction, the area enclosed by this contour has unit length in the x direction, its lower edge in the field free region $x < -s/2$ and its upper edge at the location z. Then, (1.4.1) becomes

$$\oint_C \mathbf{H}\cdot d\mathbf{s} = H_x(z) = -K_o + \int_{-s/2}^{z} J_y dz \tag{1}$$

and for $-s/2 < z < s/2$,

$$H_x = -K_o + \int_{-s/2}^{z} \frac{2J_o z}{s} dz = -K_o + \frac{J_o}{s}\left[z^2 - (s/2)^2\right] \tag{2}$$

while for $s/2 < z$,

$$H_x = 0 \tag{3}$$

1.5 CHARGE CONSERVATION IN INTEGRAL FORM

1.5.1 Because of the radial symmetry, a spherical volume having its center at the origin and a radius r is used to evaluate 1.5.2. Because the charge density is uniform, the volume integral is evaluated by simply multiplying the volume by the charge density. Thus,

$$4\pi r^2 J_r + \frac{d}{dt}\Big[\frac{4}{3}\pi r^3 \rho_o(t)\Big] = 0 \Rightarrow J_r = -\frac{r}{3}\frac{d\rho_o}{dt} \tag{1}$$

1.5.2 Equation 1.5.2 is evaluated for a volume enclosed by surfaces having area A in the planes $x = x$ and $x = 0$. Because the the current density is x directed, contributions to the surface integral over the other surfaces, which have normals that are perpendicular to the x axis, are zero. Thus, (1.5.2) becomes

$$A[J_x(x) - J_x(0)] + \frac{d}{dt}[Ax\rho_o(t)] = 0 \Rightarrow J_x = -x\frac{d\rho_o}{dt} \tag{1}$$

1.5.3 From (12),

$$\frac{\partial \sigma_s}{\partial t} = -\mathbf{n}\cdot(\mathbf{J}^a - \mathbf{J}^b) = -(0) + J_z^b(z=0) = J_o(x,y)\cos(\omega t) \tag{1}$$

Integration of this expression on time gives

$$\sigma_s = \frac{J_o(x,y)}{\omega}\sin\omega t \tag{2}$$

where the integration function of (x, y) is zero because, at every point on the surface, the surface charge density is initially zero.

1.5.4 The charge conservation continuity condition is applied to the surface at $r = R$, where $\mathbf{J}^b = 0$ and $\mathbf{n} = \mathbf{i}_r$. Thus,

$$J_o(\phi, z)\sin\omega t + \frac{\partial \sigma_s}{\partial t} = 0 \tag{1}$$

and it follows that

$$\sigma_s = -\int_0^t J_o(\phi,z)\sin\omega t dt = \frac{J_o(\phi,z)}{\omega}\cos\omega t \tag{2}$$

1.6 FARADAY'S INTEGRAL LAW

1.6.1 (a) On the contour $y = sx/g$,

$$d\mathbf{s} = dx\mathbf{i_x} + dy\mathbf{i_y} = dx\big(\mathbf{i_x} + \frac{dy}{dx}\mathbf{i_y}\big) = dx\big(\mathbf{i_x} + \frac{s}{g}\mathbf{i_y}\big) \tag{1}$$

(b) On this contour,

$$\int_a^b \mathbf{E}\cdot d\mathbf{s} = \int_0^b E_o\mathbf{i_y}\cdot\big(\mathbf{i_x} + \frac{s}{g}\mathbf{i_y}\big)dx = \int_0^g \frac{E_o s}{g}dx = E_o s \tag{2}$$

while the line integral from $(x, y) = (g, s)$ [from $b \to c$] to $(0, s)$ along $y = s$ is zero because $\mathbf{E}\cdot d\mathbf{s} = 0$. The integral over the third segment, $[c \to a]$, is

$$\int_c^a \mathbf{E}\cdot d\mathbf{s} = \int_0^s E_o\mathbf{i_y}\cdot(-\mathbf{i_y}\,dy) = -E_o s \tag{3}$$

so that

$$\oint \mathbf{E}\cdot d\mathbf{s} = E_o s - E_o s = 0 \tag{4}$$

and the circulation is indeed zero.

1.6.2 (a) The solution is as in Prob. 1.6.1 except that $dy/dx = 2sx/g^2$. Thus, the first line integral gives the same answer.

$$\int_a^b \mathbf{E}\cdot d\mathbf{s} = \int_0^g E_o\mathbf{i_y}\cdot\big(\mathbf{i_x} + \frac{2s}{g^2}x\mathbf{i_y}\big)dx = \int_0^g \frac{E_o 2sx}{g^2}dx = E_o s \tag{1}$$

Because the other contours are the same as in Prob. 1.6.1, their contributions are also the same and the net circulation is again found to be zero.

(b) The first integral is as in (b) of Prob. 1.6.2 except that the differential line element is described as in (1) and the field has the given dependence on x.

$$\int_a^b \mathbf{E}\cdot d\mathbf{s} = \int_0^g \big(\frac{E_o x}{g}\big)\big(\frac{2sx}{g^2}\big)dx = \frac{2}{3}\frac{E_o s}{g^3}x^2\Big|_0^g = \frac{2}{3}E_o s \tag{2}$$

(Note that we would now get a different answer, $E_o s/2$, if we carried out this integral using this field but the straight-line contour of Prob. 1.6.1.) From $b \to c$ there is again no contribution because $\mathbf{E}\cdot d\mathbf{s} = 0$ while from $c \to a$, the integral is

$$\int_0^s -E_o\frac{x}{g}\Big|_{x=0}dy = -\frac{E_o xy}{g}\Big|_{x=0} = 0 \tag{3}$$

which makes no contribution because the contour is at $x = 0$. Thus, the net contribution to the closed integral, the circulation, is given by (2).

1.6.3 (a) The conversion to cylindrical coordinates of (1.3.13) follows from the arguments given with the solution to Prob. 1.3.1.

$$\mathbf{E} = \frac{\lambda_l}{2\pi\epsilon_o r}\mathbf{i_r} = \frac{\lambda_l}{2\pi\epsilon_o\sqrt{x^2+y^2}}\left[\frac{x}{\sqrt{x^2+y^2}}\mathbf{i_x} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{i_y}\right] \tag{1}$$

(b) Evaluation of the line integral amounts to recognizing that on the four segments,

$$d\mathbf{s} = \mathbf{i_x}dx, \quad \mathbf{i_y}dy, \quad -\mathbf{i_x}dx, \quad -\mathbf{i_y}dy \tag{2}$$

respectively. Note that care is taken to take the endpoint of the integrals as being in the direction of an increasing coordinate. This avoids taking double account of the sign implied by the dot product $\mathbf{E}\cdot d\mathbf{s}$.

$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{s} = \frac{\lambda_l}{2\pi\epsilon_o}[\int_k^g E_x(x,0)dx \\ + \int_0^h E_y(g,y)dy + \int_k^g E_x(x,h)(-dy) + \int_0^h E_y(k,y)(-dy)\end{aligned} \tag{3}$$

These integrals become

$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{s} = \Big\{ln(g/k) + \frac{1}{2}ln(g^2+h^2) - \frac{1}{2}ln g^2 - \frac{1}{2}ln(h^2+g^2) \\ + \frac{1}{2}ln(h^2+k^2) - \frac{1}{2}ln(k^2+h^2) + \frac{1}{2}ln k^2\Big\} = 0\end{aligned} \tag{4}$$

and it follows that the sum of these contributions is indeed zero.

1.6.4 Starting at $(x,y) = (s,0)$, the line integral is

$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{s} = \int_s^d E_x(x,0)dx + \int_0^d E_y(d,y)dy - \int_0^d E_x(x,d)dx \\ - \int_s^d E_y(0,y)dy + \int_0^s E_x(x,s)dx - \int_0^s E_y(s,y)dy\end{aligned} \tag{1}$$

This expression is evaluated using $\mathbf{E}$ as given by (a) of Prob. 1.6.3 and becomes

$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{s} = \frac{\lambda_l}{2\pi\epsilon_o}\Big[\int_s^d \frac{dx}{x} + \int_0^d \frac{y}{d^2+y^2}dy - \int_0^d \frac{x}{x^2+d^2}dx \\ - \int_s^d \frac{dy}{y} + \int_0^s \frac{x}{x^2+s^2}dx - \int_0^s \frac{y}{s^2+y^2}dy\Big] = 0\end{aligned} \tag{3}$$

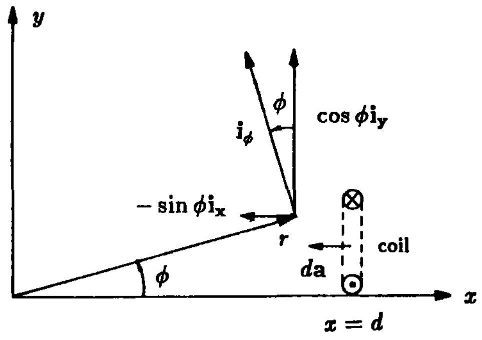

Figure S1.6.5

1.6.5 (a) In view of Fig. S1.6.5, the magnetic field given by (1.4.10)

$$\mathbf{H} = \mathbf{i}_\phi\left(\frac{i}{2\pi r}\right) \tag{1}$$

is converted to Cartesian coordinates by recognizing that

$$\mathbf{i}_\phi = -\sin\phi\mathbf{i_x} + \cos\phi\mathbf{i_y} = \frac{-y}{\sqrt{x^2+y^2}}\mathbf{i_x} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{i_y}; \quad r = \sqrt{x^2+y^2} \tag{2}$$

so that (1) becomes

$$\mathbf{H} = \frac{i}{2\pi}\left[\frac{-y}{x^2+y^2}\mathbf{i_x} + \frac{x}{x^2+y^2}\mathbf{i_y}\right] \tag{3}$$

(b) The surface of Fig. 1.7.2a, shown in terms of the $x-y$ coordinates by Fig. S1.6.5, can be used to evaluate the net flux as follows.

$$\begin{aligned}\lambda_f &= \int_S \mu_o\mathbf{H}\cdot d\mathbf{a} = l\int_0^{\sqrt{R^2-d^2}} -\mu_o H_x(d,y)dy \\ &= -\frac{l\mu_o i}{2\pi}\int_0^{\sqrt{R^2-d^2}} \frac{(-y)}{d^2+y^2}dy = \frac{\mu_o l i}{2\pi}ln(R/d)\end{aligned} \tag{4}$$

This result agrees with (1.7.5), where the flux is evaluated using a different surface. Just why the flux is the same, regardless of surface, is the point of Sec. 1.7.

(c) The circulation follows from Faraday's law, (1.6.1),

$$\oint_C \mathbf{E}\cdot d\mathbf{s} = -\frac{d\lambda_f}{dt} = -\frac{\mu_o l}{2\pi}ln(R/d)\frac{di}{dt} \tag{5}$$

(d) This flux will be linked N times by an N turn coil. Thus, the EMF at the terminals of the coil follows from (8) as

$$\mathcal{E}_{ab} = \frac{\mu_o l N}{2\pi}ln(R/d)\frac{di}{dt} \tag{6}$$

1.6.6 The left hand side of (1.6.1) is the desired circulation of $\mathbf{E}$, found by determining the right hand side, where $d\mathbf{s} = \mathbf{i}_y dxdz$.

$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{s} &= -\frac{d}{dt}\int_S \mu_o\mathbf{H}\cdot d\mathbf{s}\\ &= -\frac{d}{dt}\int_{-l/2}^{l/2}\int_0^w \mu_o H_y(x,0,z)dxdz \qquad (1)\\ &= -\mu_o wl\frac{dH_o}{dt}\end{aligned}$$

1.6.7 From (12), the tangential component of $\mathbf{E}$ must be continuous, so

$$\mathbf{n}\times(\mathbf{E}^a - \mathbf{E}^b) = 0 \Rightarrow E_x^a - E_1 = 0 \Rightarrow E_y^a = E_1 \qquad (1)$$

From (1.3.17),

$$\epsilon_o E_y^a - \epsilon_o E_2 = \sigma_o \Rightarrow E_y^a = \frac{\sigma_o}{\epsilon_o} + E_2 \qquad (2)$$

These are components of the given electric field just above the $y = 0$ surface.

1.6.8 In polar coordinates,

$$\mathbf{E} = E_o(\sin\phi\mathbf{i}_r + \cos\phi\mathbf{i}_\phi) \qquad (1)$$

The tangential component follows from (1.6.12)

$$E_\phi(r = R^+) = E_\phi(r = R^-) = E_o\cos\phi \qquad (2)$$

while the normal is given by using (1.3.17)

$$E_r(r = R^+) = \frac{\sigma_o}{\epsilon_o}\cos\phi + E_o\sin\phi \qquad (3)$$

1.7 GAUSS' INTEGRAL LAW OF MAGNETIC FLUX

1.7.1 (a) In analyzing the z directed field, note that it is perpendicular to the ϕ axis and, for $0 < \theta < \pi/2$, in the negative θ direction.

$$\mathbf{H} = H_o(\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_\theta) \qquad (1)$$

(b) Faraday's law, (1.6.1), gives the required circulation in terms of the surface integral on the right. This integral is carried out for the given surface by simply multiplying the z component of $\mathbf{H}$ by the area. The result is as given.

(c) For the hemispherical surface with its edge the same as in part (b), the normal is in the radial direction and it follows from (1) that

$$\mu_o \mathbf{H} \cdot d\mathbf{s} = (\mu_o H_o \cos\theta) r \sin\theta d\theta r d\phi \tag{2}$$

Thus, the surface integral becomes

$$\begin{aligned}\int_S \mu_o \mathbf{H} \cdot d\mathbf{a} &= \int_0^{2\pi}\int_0^{\pi/2} \mu_o H_o R^2 \cos\theta \sin\theta d\theta d\phi \\ &= \mu_o H_o R^2 (2\pi)(1/2)\end{aligned} \tag{3}$$

so that Faraday's law again gives

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\mu_o \pi R^2 \frac{dH_o}{dt} \tag{4}$$

1.7.2 The first only has contributions on the right and left surfaces, where it is of the same magnitude. Because the normals are oppositely directed on these surfaces, these integrals cancel. Thus, (a) satisfies (1.7.1).

The contributions of (b) are to the top and bottom surfaces. Because $\mathbf{H}$ differs on these two surfaces ($x = x$ on the upper surface while $x = 0$ on the lower one), this $\mathbf{H}$ has a net flux.

$$\oint_S \mathbf{H} \cdot d\mathbf{s} = \frac{AH_o x}{d} \tag{1}$$

As for (b), the top and bottom surfaces are where the only contributions can be made. This time, however, there is no net contribution because $\mathbf{H}$ does not depend on x. Thus, at each location y on the upper surface where there is a positive contribution, there is one at the same location y on the lower surface that makes a contribution of the opposite sign.

1.7.3 Continuity of the normal flux density,(1.7.6), requires that

$$\mu_0 H_y^a - \mu_o H_1 = 0 \Rightarrow H_y^a = H_1 \tag{1}$$

while Ampère's continuity condition, (1.4.16) requires that the jump in tangential $\mathbf{H}$ be equal to the given current density. Using the right hand rule,

$$H_z^a - H_2 = K_o \Rightarrow H_z^a = K_o + H_2 \tag{2}$$

These are the components of the given $\mathbf{H}$ just above the surface.

1.7.4 Given that the tangential component of $\mathbf{H}$ is zero inside the cylinder, it follows from Ampère's continuity condition, (1.4.16), that

$$H_\phi(r = R_+) = K_o \tag{1}$$

According to (1.7.6), the normal component of $\mu_o\mathbf{H}$ is continuous. Thus,

$$\mu_o H_r(r = R_+) = \mu_o H_r(r = R_-) = H_1 \tag{2}$$