Topic 7

Two- and Three-Dimensional Solid Elements; Plane Stress, Plane Strain, and Axisymmetric Conditions

Contents:

- **Isoparametric interpolations of coordinates and displacements**
- **Consistency between coordinate and displacement interpolations**
- **Meaning of these interpolations in large displacement analysis, motion of a material particle**
- **Evaluation of required derivatives**
- **The Jacobian transformations**
- **Details of strain-displacement matrices for total and updated Lagrangian formulations**
- **Example of 4-node two-dimensional element, details of matrices used**

Textbook: Sections 6.3.2, 6.3.3

Example: 6.17

- FINITE ELEMENTS CAN IN GENERAL BE CATEGORIZED AS
  - CONTINUUM ELEMENTS (SOLID)
  - STRUCTURAL ELEMENTS

IN THIS LECTURE

- WE CONSIDER THE 2-D CONTINUUM ISOPARAMETRIC ELEMENTS
- THESE ELEMENTS ARE USED VERY WIDELY
- THE ELEMENTS ARE VERY GENERAL ELEMENTS FOR GEOMETRIC AND MATERIAL NONLINEAR CONDITIONS
- WE ALSO POINT OUT HOW GENERAL 3-D ELEMENTS ARE CALCULATED USING THE SAME PROCEDURES

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TWO- AND THREE-DIMENSIONAL SOLID ELEMENTS

- Two-dimensional elements comprise
  - plane stress and plane strain elements
  - axisymmetric elements
- The derivations used for the two-dimensional elements can be easily extended to the derivation of three-dimensional elements.

  Hence we concentrate our discussion now first on the two-dimensional elements.

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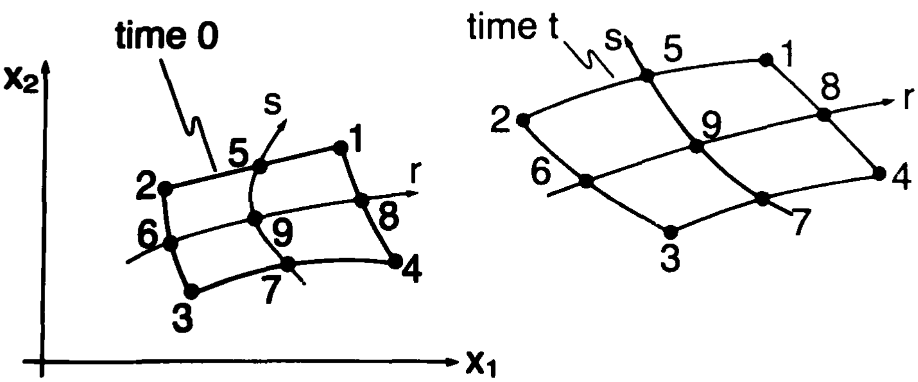
TWO-DIMENSIONAL AXISYMMETRIC, PLANE STRAIN AND PLANE STRESS ELEMENTS



Because the elements are isoparametric,

$$ {}^{0}x_1 = \sum_{k=1}^{N} h_k \, {}^{0}x_1^k \ , \quad {}^{0}x_2 = \sum_{k=1}^{N} h_k \, {}^{0}x_2^k $$

and

$$ {}^{t}x_1 = \sum_{k=1}^{N} h_k \, {}^{t}x_1^k \ , \quad {}^{t}x_2 = \sum_{k=1}^{N} h_k \, {}^{t}x_2^k $$

where the h_k's are the isoparametric interpolation functions.

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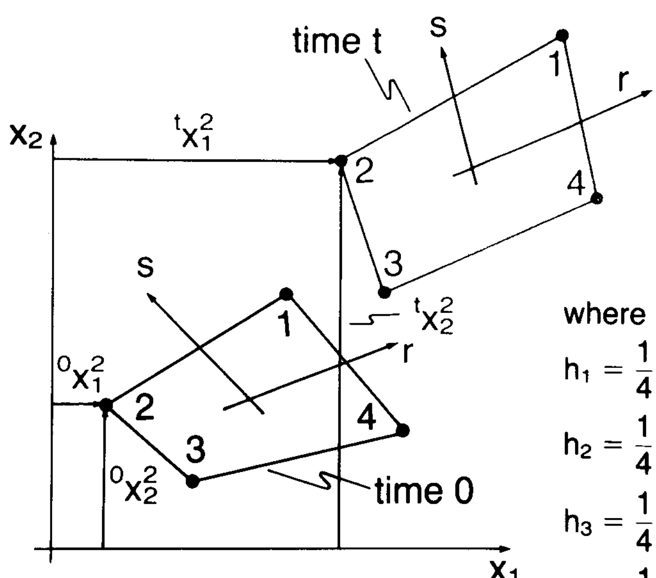
Example: A four-node element



$$ {}^{t}x_i = \sum_{k=1}^{4} h_k \, {}^{t}x_i^k $$

$$ {}^{0}x_i = \sum_{k=1}^{4} h_k \, {}^{0}x_i^k $$

where

$$ h_1 = \frac{1}{4}(1 + r)(1 + s) $$

$$ h_2 = \frac{1}{4}(1 - r)(1 + s) $$

$$ h_3 = \frac{1}{4}(1 - r)(1 - s) $$

$$ h_4 = \frac{1}{4}(1 + r)(1 - s) $$

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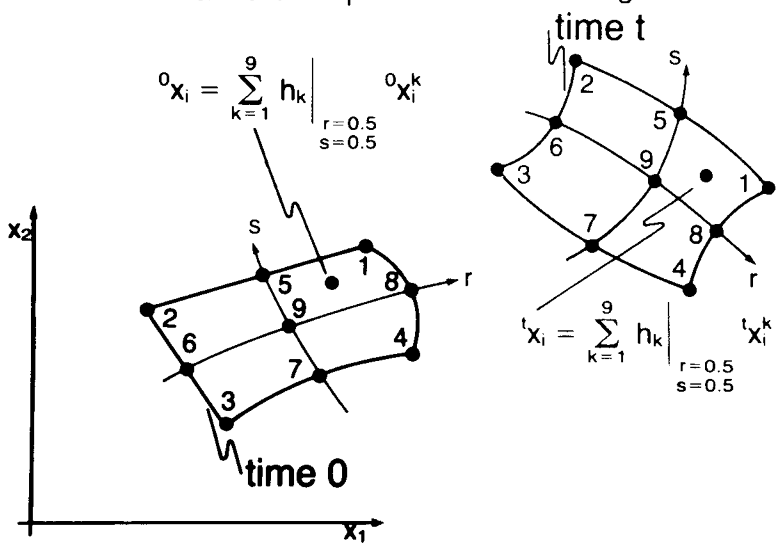
Transparency 7-5

Example: Motion of a material particle

Consider the material particle at r = 0.5, s = 0.5:

Important: The isoparametric coordinates of a material particle never change



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A major advantage of the isoparametric finite element discretization is that we may directly write

$$^tu_1 = \sum_{k=1}^{N} h_k \, {}^tu_1^k \quad , \quad {}^tu_2 = \sum_{k=1}^{N} h_k \, {}^tu_2^k$$

and

$$u_1 = \sum_{k=1}^{N} h_k \, u_1^k \quad , \quad u_2 = \sum_{k=1}^{N} h_k \, u_2^k$$

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This is easily shown: for example,

$$ {}^{t}x_i = \sum_{k=1}^{N} h_k \, {}^{t}x_i^k $$

$$ {}^{0}x_i = \sum_{k=1}^{N} h_k \, {}^{0}x_i^k $$

Subtracting the second equation from the first equation gives

$$ \underbrace{{}^{t}x_i - {}^{0}x_i}_{{}^{t}u_i} = \sum_{k=1}^{N} h_k \, \underbrace{({}^{t}x_i^k - {}^{0}x_i^k)}_{{}^{t}u_i^k} $$

Transparency 7-8

The element matrices require the following derivatives:

$$ {}_{0}^{t}u_{i,j} = \frac{\partial\, {}^{t}u_i}{\partial\, {}^{0}x_j} = \sum_{k=1}^{N} \left(\frac{\partial h_k}{\partial\, {}^{0}x_j} \right) {}^{t}u_i^k $$

$$ {}_{0}u_{i,j} = \frac{\partial u_i}{\partial\, {}^{0}x_j} = \sum_{k=1}^{N} \left(\frac{\partial h_k}{\partial\, {}^{0}x_j} \right) u_i^k $$

$$ {}_{t}u_{i,j} = \frac{\partial u_i}{\partial\, {}^{t}x_j} = \sum_{k=1}^{N} \left(\frac{\partial h_k}{\partial\, {}^{t}x_j} \right) u_i^k $$

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These derivatives are evaluated using a Jacobian transformation (the chain rule):

$$\frac{\partial h_k}{\partial r} = \boxed{\frac{\partial h_k}{\partial\, {}^0x_1}}\, \frac{\partial\, {}^0x_1}{\partial r} + \boxed{\frac{\partial h_k}{\partial\, {}^0x_2}}\, \frac{\partial\, {}^0x_2}{\partial r}$$

$$\frac{\partial h_k}{\partial s} = \boxed{\frac{\partial h_k}{\partial\, {}^0x_1}}\, \frac{\partial\, {}^0x_1}{\partial s} + \boxed{\frac{\partial h_k}{\partial\, {}^0x_2}}\, \frac{\partial\, {}^0x_2}{\partial s}$$

REQUIRED DERIVATIVES

In matrix form,

$$\begin{bmatrix} \dfrac{\partial h_k}{\partial r} \\[2ex] \dfrac{\partial h_k}{\partial s} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial\, {}^0x_1}{\partial r} & \dfrac{\partial\, {}^0x_2}{\partial r} \\[2ex] \dfrac{\partial\, {}^0x_1}{\partial s} & \dfrac{\partial\, {}^0x_2}{\partial s} \end{bmatrix}}_{{}^0\underline{J}} \begin{bmatrix} \dfrac{\partial h_k}{\partial\, {}^0x_1} \\[2ex] \dfrac{\partial h_k}{\partial\, {}^0x_2} \end{bmatrix}$$

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The required derivatives are computed using a matrix inversion:

$$\begin{bmatrix} \dfrac{\partial h_k}{\partial\, {}^0x_1} \\[2ex] \dfrac{\partial h_k}{\partial\, {}^0x_2} \end{bmatrix} = {}^0\underline{J}^{-1} \begin{bmatrix} \dfrac{\partial h_k}{\partial r} \\[2ex] \dfrac{\partial h_k}{\partial s} \end{bmatrix}$$

The entries in ${}^0\underline{J}$ are computed using the interpolation functions. For example,

$$\frac{\partial\, {}^0x_1}{\partial r} = \sum_{k=1}^{N} \frac{\partial h_k}{\partial r}\, {}^0x_1^k$$

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The derivatives taken with respect to the configuration at time t can also be evaluated using a Jacobian transformation.

$$\begin{bmatrix} \dfrac{\partial h_k}{\partial r} \\[2mm] \dfrac{\partial h_k}{\partial s} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial\, {}^t x_1}{\partial r} & \dfrac{\partial\, {}^t x_2}{\partial r} \\[2mm] \dfrac{\partial\, {}^t x_1}{\partial s} & \boxed{\dfrac{\partial\, {}^t x_2}{\partial s}} \end{bmatrix}}_{{}^t\underline{J}} \begin{bmatrix} \dfrac{\partial h_k}{\partial\, {}^t x_1} \\[2mm] \dfrac{\partial h_k}{\partial\, {}^t x_2} \end{bmatrix}$$

$$\sum_{k=1}^{N} \frac{\partial h_k}{\partial s}\, {}^t x_2^k$$

$$\begin{bmatrix} \dfrac{\partial h_k}{\partial\, {}^t x_1} \\[2mm] \dfrac{\partial h_k}{\partial\, {}^t x_2} \end{bmatrix} = {}^t\underline{J}^{-1} \begin{bmatrix} \dfrac{\partial h_k}{\partial r} \\[2mm] \dfrac{\partial h_k}{\partial s} \end{bmatrix}$$

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We can now compute the required element matrices for the total Lagrangian formulation:

Element Matrix	Matrices Required
${}_0^t\underline{K}_L$	${}_0\underline{C}$, ${}_0^t\underline{B}_L$
${}_0^t\underline{K}_{NL}$	${}_0^t\underline{S}$, ${}_0^t\underline{B}_{NL}$
${}_0^t\underline{F}$	${}_0^t\hat{\underline{S}}$, ${}_0^t\underline{B}_L$

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We define ${}_0\underline{C}$ so that

$$\begin{bmatrix} {}_0S_{11} \\ {}_0S_{22} \\ {}_0S_{12} \\ {}_0S_{33} \end{bmatrix} = {}_0\underline{C} \begin{bmatrix} {}_0e_{11} \\ {}_0e_{22} \\ 2\,{}_0e_{12} \\ {}_0e_{33} \end{bmatrix}$$

analogous to
${}_0S_{ij} = {}_0C_{ijrs}\,{}_0e_{rs}$

For example, we may choose (axisymmetric analysis),

$${}_0\underline{C} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$$

Transparency 7-14

We note that, in two-dimensional analysis,

$${}_0e_{11} = {}_0u_{1,1} + \underbrace{{}^t_0u_{1,1}\;{}_0u_{1,1} + {}^t_0u_{2,1}\;{}_0u_{2,1}}$$

$${}_0e_{22} = {}_0u_{2,2} + \underbrace{{}^t_0u_{1,2}\;{}_0u_{1,2} + {}^t_0u_{2,2}\;{}_0u_{2,2}}$$

$$2\,{}_0e_{12} = ({}_0u_{1,2} + {}_0u_{2,1}) + \underbrace{({}^t_0u_{1,1}\,{}_0u_{1,2} + {}^t_0u_{2,1}\,{}_0u_{2,2} + {}^t_0u_{1,2}\,{}_0u_{1,1} + {}^t_0u_{2,2}\,{}_0u_{2,1})}$$

$${}_0e_{33} = \frac{u_1}{{}^0x_1} + \underbrace{\left(\frac{{}^tu_1}{{}^0x_1}\right)\frac{u_1}{{}^0x_1}}$$

INITIAL DISPLACEMENT EFFECT

Transparency 7-15

and

$$_0\eta_{11} = \frac{1}{2}\left(({}_0u_{1,1})^2 + ({}_0u_{2,1})^2\right)$$

$$_0\eta_{22} = \frac{1}{2}\left(({}_0u_{1,2})^2 + ({}_0u_{2,2})^2\right)$$

$$_0\eta_{12} = {}_0\eta_{21} = \frac{1}{2}\left({}_0u_{1,1}\ {}_0u_{1,2} + {}_0u_{2,1}\ {}_0u_{2,2}\right)$$

$$_0\eta_{33} = \frac{1}{2}\left(\frac{u_1}{{}^0x_1}\right)^2$$

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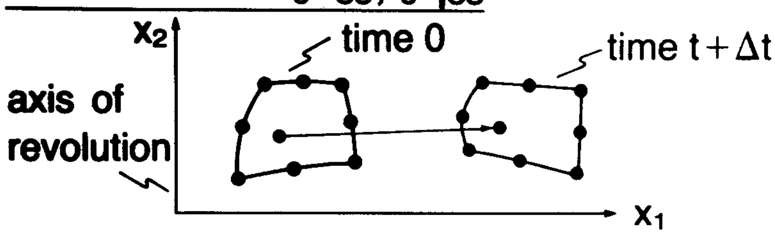
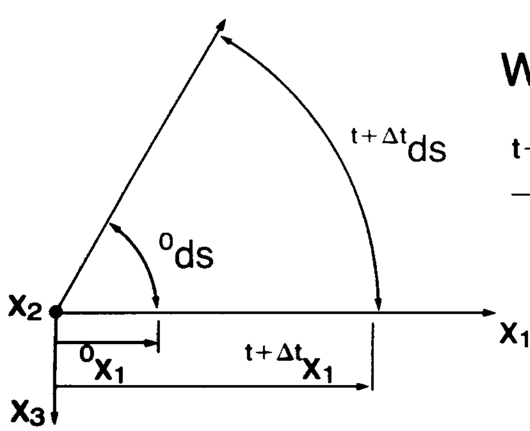
Derivation of $_0e_{33}$, $_0\eta_{33}$:





We see that

$$\frac{{}^{t+\Delta t}ds}{{}^0ds} = \frac{{}^{t+\Delta t}x_1}{{}^0x_1}$$

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Hence

$$
\begin{aligned}
{}^{t+\Delta t}_{0}\varepsilon_{33} &= \frac{1}{2}\left[\left(\frac{{}^{t+\Delta t}ds}{{}^{0}ds}\right)^2 - 1\right] \\
&= \frac{1}{2}\left[\left(\frac{{}^{t+\Delta t}x_1}{{}^{0}x_1}\right)^2 - 1\right] \\
&= \frac{1}{2}\left[\left(\frac{{}^{0}x_1 + {}^{t}u_1 + u_1}{{}^{0}x_1}\right)^2 - 1\right] \\
&\;\vdots \\
&= \underbrace{\left(\frac{{}^{t}u_1}{{}^{0}x_1} + \frac{1}{2}\left(\frac{{}^{t}u_1}{{}^{0}x_1}\right)^2\right)}_{{}^{t}_{0}\varepsilon_{33}} \\
&\quad + \underbrace{\left(\underbrace{\frac{u_1}{{}^{0}x_1} + \left(\frac{{}^{t}u_1}{{}^{0}x_1}\right)\frac{u_1}{{}^{0}x_1}}_{{}_{0}e_{33}} + \underbrace{\frac{1}{2}\left(\frac{u_1}{{}^{0}x_1}\right)^2}_{{}_{0}\eta_{33}}\right)}_{{}_{0}\varepsilon_{33}}
\end{aligned}
$$

Transparency 7-18

We construct ${}^{t}_{0}\underline{B}_{L}$ so that

$$
\begin{bmatrix} {}_{0}e_{11} \\ {}_{0}e_{22} \\ 2\,{}_{0}e_{12} \\ \hline {}_{0}e_{33} \end{bmatrix} = {}_{0}\underline{e} = \underbrace{({}^{t}_{0}\underline{B}_{L0} + {}^{t}_{0}\underline{B}_{L1})}_{{}^{t}_{0}\underline{B}_{L}}\ \hat{\underline{u}}
$$

${}_{0}e_{33}$ is only included for axisymmetric analysis

contains initial displacement effect

Entries in ${}^t_0\underline{B}_{L0}$:

$$
\begin{array}{c}
\overbrace{\begin{array}{cc} \quad u_1^k \quad & \quad u_2^k \quad \end{array}}^{\text{node k}} \\
\left[\begin{array}{c|cc|c}
 & {}_0h_{k,1} & 0 & \\
\cdots & 0 & {}_0h_{k,2} & \cdots \\
 & {}_0h_{k,2} & {}_0h_{k,1} & \\
\hline
 & h_k/{}^0x_1 & 0 &
\end{array}\right]
\left[\begin{array}{c} \vdots \\ \hline u_1^k \\ u_2^k \\ \hline \vdots \end{array}\right]
\begin{array}{c} \\ \uparrow \\ \text{node k} \\ \downarrow \\ \end{array}
\end{array}
$$

included only for axisymmetric analysis

This is similar in form to the $\underline{B}$ matrix used in linear analysis.

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Entries in ${}^t_0\underline{B}_{L1}$:

$$
\begin{array}{c}
\overbrace{\begin{array}{cc} \qquad u_1^k \qquad & \qquad u_2^k \qquad \end{array}}^{\text{node k}} \\
\left[\begin{array}{c|c|c|c}
 & {}^t_0u_{1,1}\ {}_0h_{k,1} & {}^t_0u_{2,1}\ {}_0h_{k,1} & \\
\cdots & {}^t_0u_{1,2}\ {}_0h_{k,2} & {}^t_0u_{2,2}\ {}_0h_{k,2} & \cdots \\
 & {}^t_0u_{1,1}\ {}_0h_{k,2} & {}^t_0u_{2,1}\ {}_0h_{k,2} & \\
 & +\ {}^t_0u_{1,2}\ {}_0h_{k,1} & +\ {}^t_0u_{2,2}\ {}_0h_{k,1} & \\
\hline
 & \dfrac{{}^tu_1}{{}^0x_1}\ \dfrac{h_k}{{}^0x_1} & 0 &
\end{array}\right]
\end{array}
$$

included only for axisymmetric analysis

The initial displacement effect is contained in the terms ${}^t_0u_{i,j}$, $\dfrac{{}^tu_1}{{}^0x_1}$.

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Transparency 7-21

We construct ${}_0^t\underline{B}_{NL}$ and ${}_0^t\underline{S}$ so that

$$\delta\hat{\underline{u}}^T\, {}_0^t\underline{B}_{NL}^T\, {}_0^t\underline{S}\, {}_0^t\underline{B}_{NL}\, \hat{\underline{u}} = {}_0^tS_{ij}\, \delta_0\eta_{ij}$$

Entries in ${}_0^t\underline{S}$:

$$\left[\begin{array}{cccc|c} {}_0^tS_{11} & {}_0^tS_{12} & 0 & 0 & 0 \\ {}_0^tS_{21} & {}_0^tS_{22} & 0 & 0 & 0 \\ 0 & 0 & {}_0^tS_{11} & {}_0^tS_{12} & 0 \\ 0 & 0 & {}_0^tS_{21} & {}_0^tS_{22} & 0 \\ \hline 0 & 0 & 0 & 0 & {}_0^tS_{33} \end{array}\right]$$

included only for axisymmetric analysis

Transparency 7-22

Entries in ${}_0^t\underline{B}_{NL}$:

$$\begin{array}{c} \overbrace{\begin{array}{cc} u_1^k & u_2^k \end{array}}^{\text{node } k} \\ \left[\begin{array}{c|c|c|c} & {}_0h_{k,1} & 0 & \\ & {}_0h_{k,2} & 0 & \\ \cdots & 0 & {}_0h_{k,1} & \cdots \\ & 0 & {}_0h_{k,2} & \\ \hline & h_k/{}^0x_1 & 0 & \end{array}\right] \end{array} \begin{bmatrix} \vdots \\ \hline u_1^k \\ u_2^k \\ \hline \vdots \end{bmatrix} \begin{array}{c} \\ \text{node } k \\ \\ \end{array}$$

included only for axisymmetric analysis

${}_0^t\hat{\underline{S}}$ is constructed so that

$$\delta \hat{\underline{u}}^T \; {}_0^t\underline{B}_L^T \; {}_0^t\hat{\underline{S}} = {}_0^tS_{ij} \, \delta_0 e_{ij}$$

Entries in ${}_0^t\hat{\underline{S}}$:

$$\begin{bmatrix} {}_0^tS_{11} \\ {}_0^tS_{22} \\ {}_0^tS_{12} \\ \hline {}_0^tS_{33} \end{bmatrix}$$

included only for axisymmetric analysis

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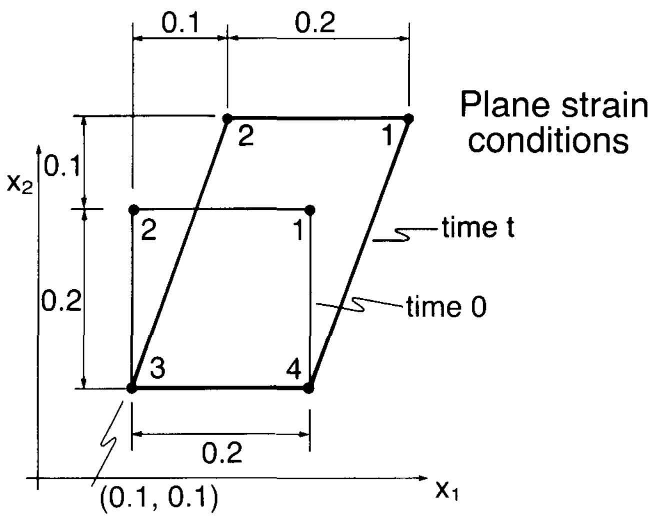
Example: Calculation of ${}_0^t\underline{B}_L$, ${}_0^t\underline{B}_{NL}$



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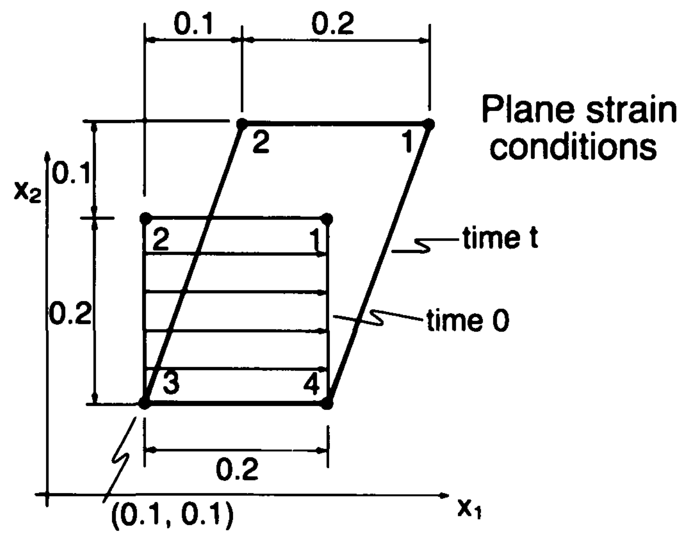
Example: Calculation of ${}^t_0\underline{B}_L$, ${}^t_0\underline{B}_{NL}$



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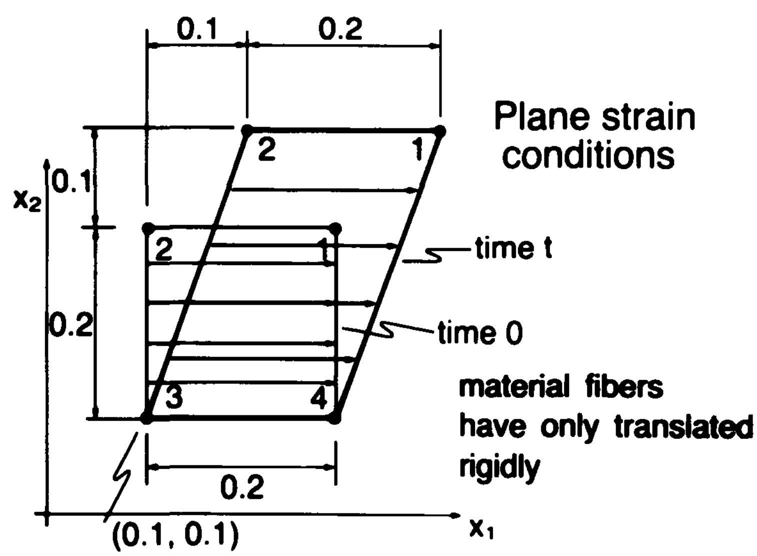
Example: Calculation of ${}^t_0\underline{B}_L$, ${}^t_0\underline{B}_{NL}$



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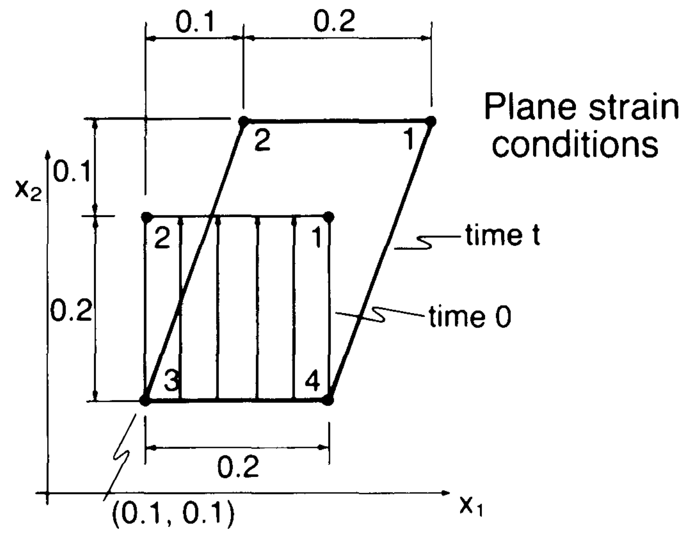
Example: Calculation of ${}^{t}_{0}\underline{B}_{L}$, ${}^{t}_{0}\underline{B}_{NL}$



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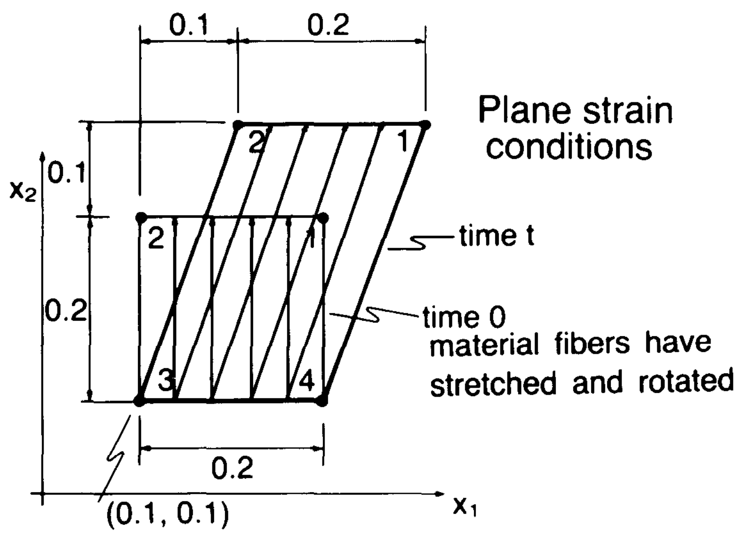
Example: Calculation of ${}^{t}_{0}\underline{B}_{L}$, ${}^{t}_{0}\underline{B}_{NL}$



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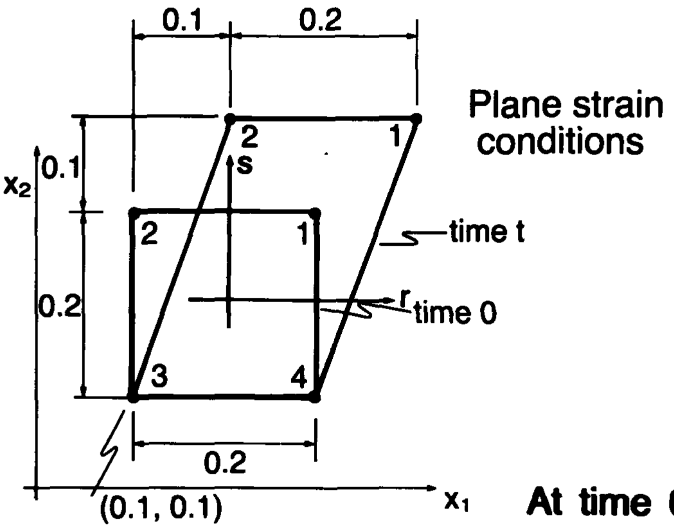
Example: Calculation of ${}_0^t\underline{B}_L$, ${}_0^t\underline{B}_{NL}$



At time 0,

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We can now perform a Jacobian transformation between the (r, s) coordinate system and the $({}^0x_1, {}^0x_2)$ coordinate system:

By inspection, $\frac{\partial {}^0x_1}{\partial r} = 0.1\ , \frac{\partial {}^0x_2}{\partial r} = 0$

$$\frac{\partial {}^0x_1}{\partial s} = 0 \quad , \frac{\partial {}^0x_2}{\partial s} = 0.1$$

Hence ${}^0\underline{J} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $|{}^0\underline{J}| = 0.01$

and

$$\frac{\partial}{\partial {}^0x_1} = 10\,\frac{\partial}{\partial r}\ , \quad \frac{\partial}{\partial {}^0x_2} = 10\,\frac{\partial}{\partial s}$$

Now we use the interpolation functions to compute ${}_0^t u_{1,1}$, ${}_0^t u_{1,2}$:

node k	$\frac{\partial h_k}{\partial {}^0x_1}$	$\frac{\partial h_k}{\partial {}^0x_2}$	${}^t u_1^k$	$\frac{\partial h_k}{\partial {}^0x_1} {}^t u_1^k$	$\frac{\partial h_k}{\partial {}^0x_2} {}^t u_1^k$
1	2.5(1 + s)	2.5(1 + r)	0.1	0.25(1 + s)	0.25(1 + r)
2	−2.5(1 + s)	2.5(1 − r)	0.1	−0.25(1 + s)	0.25(1 − r)
3	−2.5(1 − s)	−2.5(1 − r)	0.0	0	0
4	2.5(1 − s)	−2.5(1 + r)	0.0	0	0

Sum: $\underbrace{0.0}_{{}_0^t u_{1,1}}$ $\underbrace{0.5}_{{}_0^t u_{1,2}}$

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For this simple problem, we can compute the displacement derivatives by inspection:

From the given dimensions,

$$
{}_0^t\underline{X} = \begin{bmatrix} 1.0 & 0.5 \\ 0.0 & 1.5 \end{bmatrix}
$$

Hence

$$
\begin{aligned}
{}_0^t u_{1,1} &= {}_0^t X_{11} - 1 = 0 \\
{}_0^t u_{1,2} &= {}_0^t X_{12} \quad\;\; = 0.5 \\
{}_0^t u_{2,1} &= {}_0^t X_{21} \quad\;\; = 0 \\
{}_0^t u_{2,2} &= {}_0^t X_{22} - 1 = 0.5
\end{aligned}
$$

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Transparency 7-33

We can now construct the columns in ${}_0^t\underline{B}_L$ that correspond to node 3:

$$\left[\begin{array}{c|c:c|c} \cdots & -2.5(1-s) & 0 & \cdots \\ & 0 & -2.5(1-r) & \\ & -2.5(1-r) & -2.5(1-s) & \end{array}\right] \quad {}_0^t\underline{B}_{L0}$$

$$\left[\begin{array}{c|c:c|c} \cdots & 0 & 0 & \cdots \\ & -1.25(1-r) & -1.25(1-r) & \\ & -1.25(1-s) & -1.25(1-s) & \end{array}\right] \quad {}_0^t\underline{B}_{L1}$$

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Similarly, we construct the columns in ${}_0^t\underline{B}_{NL}$ that correspond to node 3:

$$\left[\begin{array}{c|c:c|c} \cdots & -2.5(1-s) & 0 & \cdots \\ & -2.5(1-r) & 0 & \\ & 0 & -2.5(1-s) & \\ & 0 & -2.5(1-r) & \end{array}\right]$$

Transparency 7-35

Consider next the element matrices required for the updated Lagrangian formulation:

Element Matrix	Matrices Required
${}^t_t\underline{K}_L$	${}_t\underline{C}$, ${}^t_t\underline{B}_L$
${}^t_t\underline{K}_{NL}$	${}^t\underline{\tau}$, ${}^t_t\underline{B}_{NL}$
${}^t_t\underline{F}$	${}^t\underline{\hat{\tau}}$, ${}^t_t\underline{B}_L$

Transparency 7-36

We define ${}_t\underline{C}$ so that

$$\begin{bmatrix} {}_tS_{11} \\ {}_tS_{22} \\ {}_tS_{12} \\ {}_tS_{33} \end{bmatrix} = {}_t\underline{C} \begin{bmatrix} {}_te_{11} \\ {}_te_{22} \\ 2\,{}_te_{12} \\ {}_te_{33} \end{bmatrix}$$

analogous to ${}_tS_{ij} = {}_tC_{ijrs}\,{}_te_{rs}$

For example, we may choose (axisymmetric analysis),

$${}_t\underline{C} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$$

Transparency 7-37

We note that the incremental strain components are, in two-dimensional analysis,

$$
\begin{aligned}
{}_te_{11} &= \frac{\partial u_1}{\partial\, {}^tx_1} = {}_tu_{1,1} \\
{}_te_{22} &= {}_tu_{2,2} \\
2\, {}_te_{12} &= {}_tu_{1,2} + {}_tu_{2,1} \\
{}_te_{33} &= u_1 / {}^tx_1
\end{aligned}
$$

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and

$${}_t\eta_{11} = \frac{1}{2}\left(({}_tu_{1,1})^2 + ({}_tu_{2,1})^2\right)$$

$${}_t\eta_{22} = \frac{1}{2}\left(({}_tu_{1,2})^2 + ({}_tu_{2,2})^2\right)$$

$${}_t\eta_{12} = {}_t\eta_{21} = \frac{1}{2}\left({}_tu_{1,1}\, {}_tu_{1,2} + {}_tu_{2,1}\, {}_tu_{2,2}\right)$$

$${}_t\eta_{33} = \frac{1}{2}\left(\frac{u_1}{{}^tx_1}\right)^2$$

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We construct ${}^{t}_{t}\underline{B}_L$ so that

$$\begin{bmatrix} {}_t e_{11} \\ {}_t e_{22} \\ 2\,{}_t e_{12} \\ \hline {}_t e_{33} \end{bmatrix} = {}_t\underline{e} = {}^{t}_{t}\underline{B}_L\, \underline{\hat{u}}$$

${}_t e_{33}$: only included for axisymmetric analysis

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Entries in ${}^{t}_{t}\underline{B}_L$:

$$\begin{array}{c} \overbrace{\begin{array}{cc} u_1^k & u_2^k \end{array}}^{\text{node } k} \\ \begin{bmatrix} \cdots & {}_t h_{k,1} & 0 & \cdots \\ & 0 & {}_t h_{k,2} & \\ & {}_t h_{k,2} & {}_t h_{k,1} & \\ \hline & h_k/{}^t x_1 & 0 & \end{bmatrix} \end{array} \begin{bmatrix} \vdots \\ u_1^k \\ u_2^k \\ \vdots \end{bmatrix} \begin{array}{l} \\ \} \text{ node } k \\ \\ \end{array}$$

only included for axisymmetric analysis

This is similar in form to the $\underline{B}$ matrix used in linear analysis.

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We construct ${}^t_t\underline{B}_{NL}$ and ${}^t\underline{\tau}$ so that

$$\delta\underline{\hat{u}}^T\,{}^t_t\underline{B}_{NL}^T\,{}^t\underline{\tau}\,{}^t_t\underline{B}_{NL}\,\underline{\hat{u}} = {}^t\tau_{ij}\,\delta_t\eta_{ij}$$

Entries in ${}^t\underline{\tau}$:

$$\left[\begin{array}{cccc|c}
{}^t\tau_{11} & {}^t\tau_{12} & 0 & 0 & 0 \\
{}^t\tau_{21} & {}^t\tau_{22} & 0 & 0 & 0 \\
0 & 0 & {}^t\tau_{11} & {}^t\tau_{12} & 0 \\
0 & 0 & {}^t\tau_{21} & {}^t\tau_{22} & 0 \\
\hline
0 & 0 & 0 & 0 & {}^t\tau_{33}
\end{array}\right]$$

included only for axisymmetric analysis

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Entries in ${}^t_t\underline{B}_{NL}$:

$$\begin{array}{c}
\text{node } k \\
\begin{array}{cc} u_1^k & u_2^k \end{array}
\end{array}$$

$$\left[\begin{array}{c|c|c|c}
 & {}_th_{k,1} & 0 & \\
 & {}_th_{k,2} & 0 & \\
\cdots & 0 & {}_th_{k,1} & \cdots \\
 & 0 & {}_th_{k,2} & \\
\hline
 & h_k/{}^tx_1 & 0 &
\end{array}\right]
\quad
\left[\begin{array}{c}
\vdots \\
\hline
u_1^k \\
u_2^k \\
\hline
\\
\end{array}\right]
\begin{array}{l} \\ \text{node } k \\ \\ \end{array}$$

included only for axisymmetric analysis

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${}^t\hat{\underline{T}}$ is constructed so that

$$\delta \hat{\underline{u}}^T \; {}^t_t\underline{B}_L^T \; {}^t\hat{\underline{T}} = {}^t\tau_{ij} \, \delta_t e_{ij}$$

Entries in ${}^t\hat{\underline{T}}$:

$$\begin{bmatrix} {}^t\tau_{11} \\ {}^t\tau_{22} \\ {}^t\tau_{12} \\ \hline {}^t\tau_{33} \end{bmatrix}$$

included only for axisymmetric analysis

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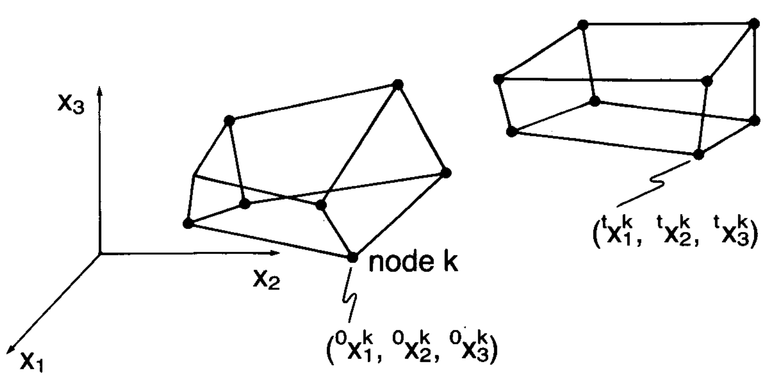
Three-dimensional elements



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Here we now use

$$ {}^{0}x_1 = \sum_{k=1}^{N} h_k \, {}^{0}x_1^k \ , \quad {}^{0}x_2 = \sum_{k=1}^{N} h_k \, {}^{0}x_2^k $$

$$ {}^{0}x_3 = \sum_{k=1}^{N} h_k \, {}^{0}x_3^k \ , $$

where the h_k's are the isoparametric interpolation functions of the three-dimensional element.

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Also

$$ {}^{t}x_1 = \sum_{k=1}^{N} h_k \, {}^{t}x_1^k \ , \quad {}^{t}x_2 = \sum_{k=1}^{N} h_k \, {}^{t}x_2^k $$

$$ {}^{t}x_3 = \sum_{k=1}^{N} h_k \, {}^{t}x_3^k $$

and then all the concepts and derivations already discussed are directly applicable to the derivation of the three-dimensional element matrices.