Unit 6: Conformal Mapping

1. Overview

In the last exercise (optional) of the previous unit, we showed that if f was analytic at $z = z_0$ (i.e., in a neighborhood of z_0) and if $f'(z_0) \neq 0$ then f mapped sufficiently small neighborhoods of z_0 conformally into neighborhoods of $w_0 = f(z_0)$. That is, the region R in the z-plane was mapped into the region S in the w-plane in such a way that S was "essentially" the same shape as R (but perhaps magnified and rotated). We pointed out that this conformal mapping, that could be identified with f, helped us visualize the properties of f just as in the real case when we often studied the properties of a function in terms of its graph.

In addition, conformal mapping comes up in a rather natural way in <u>many</u> branches of mathematical physics; and this alone would explain the immediate usefulness of complex function theory in the "real world." Moreover, because of this wide application to the real world, it is understandable that all questions concerning conformal mappings has received a great deal of attention (both from the applied and the pure mathematicians as well as other scientists and engineers).

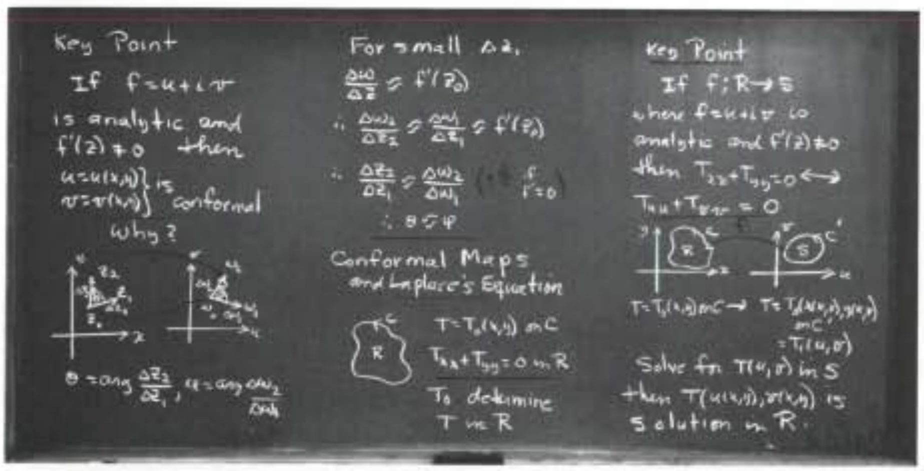
Because of the nature of our course, we do not begin to pursue this topic in any great detail (at least relative to what is usually called "great detail") but rather we simply try to give a general introduction to the topic in terms of it being a natural extension of our discussion about analytic functions.

This unit, as usual, begins with a lecture; and the lecture is designed to give you a rather quick insight to what a conformal mapping really is. The remainder of the unit is then devoted to having you try your hand at the exercises, all of which have been designated as learning exercises if only because the treatment of conformal mappings is not discussed at all in the text.

We have elected <u>not</u> to supplement this unit with our own notes, based solely on the judgment that there is really no brief way of talking meaningfully about the details that arise in the study of conformal mappings without presenting a much more thorough course in complex variables than is appropriate for our purposes.

2. Lecture 1.030

a.



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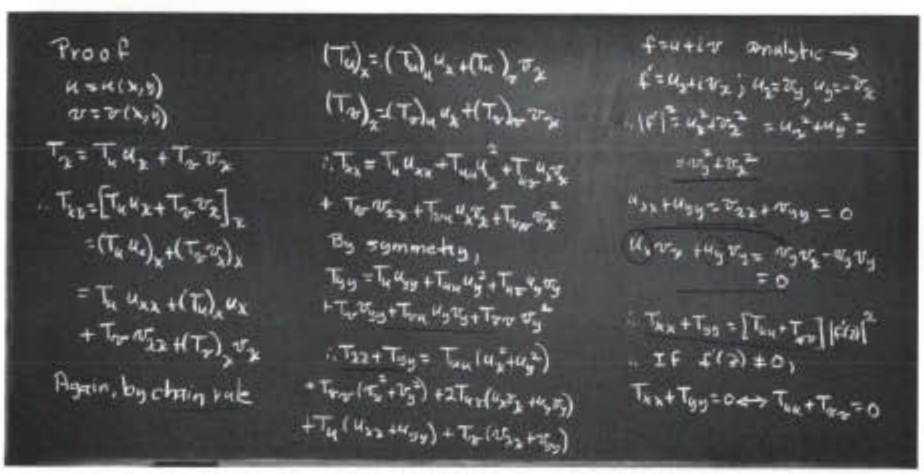


c.

3. Exercises:

1.6.1(L)

Let $f:C\to C$ (where C denotes the complex number system) be defined by $f(z) = z^2$.

a. Describe the region $S = f(R)$ if R is a "sufficiently small" region of the z-plane, centered at $z = i$.

b. Let R be the rectangle centered at $z = i$ with vertices at the points $A(-h,1-2h)$, $B(h,1-2h)$, $C(h,1+2h)$, $D(-h,1+2h)$ where h is an arbitrarily chosen positive (real) number. [Note: As a reminder, recall that the point (a,b) in the z-plane (Argand Diagram) means the complex number, $a + bi$.] Sketch the region $S = f(R)$.

c. Explain why the result of part (b) does not contradict the result of part (a).

d. Let R be a sufficiently small region centered in the z-plane at $z = 1 + i$. Describe $S = f(R)$.

1.6.2(L)

Again, let $f(z) = z^2$. Let S_1 be the line (ray) $\theta = C°$ in the z-plane, and let S_2 be the ray obtained by rotating S_1 through $+90°$ [i.e., S_2 is the ray $\theta = C° + 90°$.] Describe the image, under f, of $\measuredangle AOB$ where A is a point on S_1, B is a point on S_2, and O is the origin (i.e., $z = 0$).

1.6.3(L)

Let $f(z) = \bar{z}$.

a. Describe the image S of any region R in the z-plane with respect to the mapping f. In particular, discuss, relative to R, the size, shape, and orientation of S.

b. Use the result of part (a) to show that a 1-1 mapping need not be conformal.

c. [Optional in the sense that rehashes, but a bit more slowly, a point discussed in the lecture.] Show that if f analytic at z_0 and $f'(z_0) \neq 0$ (i.e., f is conformal at z_0) then f must be 1-1 in sufficiently small neighborhoods of $z = z_0$.

1.6.4(L)

Let $f(z) = az + b$, where a and b are complex numbers.

a. Use the value of f' to prove that f is conformal.

b. With a and b as in part (a), write $a = a_1 + ia_2$ and $b = b_1 + ib_2$ to prove that the linear mapping defined by

$$u = a_1x - a_2y + b_1$$

$$v = a_2x + a_1y + b_2$$

is conformal, provided only that a_1 and a_2 are not <u>both</u> 0.

c. Use the results of (b) to describe the mapping of the xy-plane into the uv-plane defined by

$$\left.\begin{array}{l} u = 3x - 2y + 5 \\ v = 2x + 3y + 12 \end{array}\right\} .$$

1.6.5(L)

a. Suppose $f = u + iv$ where

$$\begin{cases} u = 3x + 2y \\ v = x + y \end{cases} .$$

Write f(z) explicitly in the form $f(z) = az + b\bar{z}$ where a and b are suitably chosen complex constants. In particular, determine the values of a and b.

b. Use the result of part (a) to deduce that f is not conformal.

c. If $u = 3x - 2y + 5$, how must v be chosen if $u + iv$ is to be conformal?

1.6.6(L)

Suppose we map the xy-plane (excluding the origin) into the uv-plane by the rule:

(continued on next page)

1.6.6(L) continued

$$\left.\begin{aligned} u &= \frac{x}{x^2 + y^2} \\ v &= -\frac{y}{x^2 + y^2} \end{aligned}\right\}$$

[This is why (0,0) is excluded. Namely, both u and v have the indeterminate form $\frac{0}{0}$ when $x = y = 0$.]

Show that in the neighborhood of any point (other than the origin) this mapping is conformal.

1.6.7(L)

Let $f = u + iv$ be defined by

$$\begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}.$$

a. Show that f is analytic.

b. Show that f is conformal in sufficiently small neighborhoods of every point in the z-plane. [In particular, this explains why the curves $e^x \cos y$ = constant intersect the curves $e^x \sin y$ = constant at right angles (i.e., $u = C_1$ and $v = C_2$ are orthogonal because $x = C_1$ and $y = C_2$ are).]

c. [Optional - This reinforces the latter part of our lecture in this unit and may be omitted if you feel that the point was adequately described in the lecture. Our main point is to try to indicate what it means physically to map one region conformally onto another.]

Show that with u and v as defined above that if T is any twice differentiable function of x and y then

$$T_{xx} + T_{yy} = e^{2x}(T_{uu} + T_{vv}).$$

Preface to the Next Exercises:

The remaining exercises are all optional. They concentrate on a rather special class of conformal mappings - a class which is easy enough for us to discuss in a meaningful way without having to know too much theory about complex functions. In particular, these exercises discuss class of functions known as the linear group. These functions map circles and lines into circles and lines and consequently they are convenient to describe.

While the more general theory is beyond our scope, the results are worth mentioning. Namely, a rather famous theorem known as the Reimann Mapping Theory says essentially that if R is any simply-connected region which is a proper subset of the z-plane (i.e., we do not allow R to be the entire z-plane) and if z_o is a prescribed point in R then there is a unique analytic function f such that $f(z_o) = 0$ and $f'(z_o) > 0$ which maps R onto the unit disc $|w| < 1$ in a 1-1 manner. [The condition that $f'(z_o) > 0$ merely means that if $f'(z)$ is not greater than 0, for example, if $f'(z)$ is either negative or non-real, we may multiply f by a suitable constant so that the new function has its derivative positive at z_o.]

The key point is that the Riemann Mapping Theorem allows us to map a simply-connected region R conformally onto a given simply-connected region S in a unique way as follows:

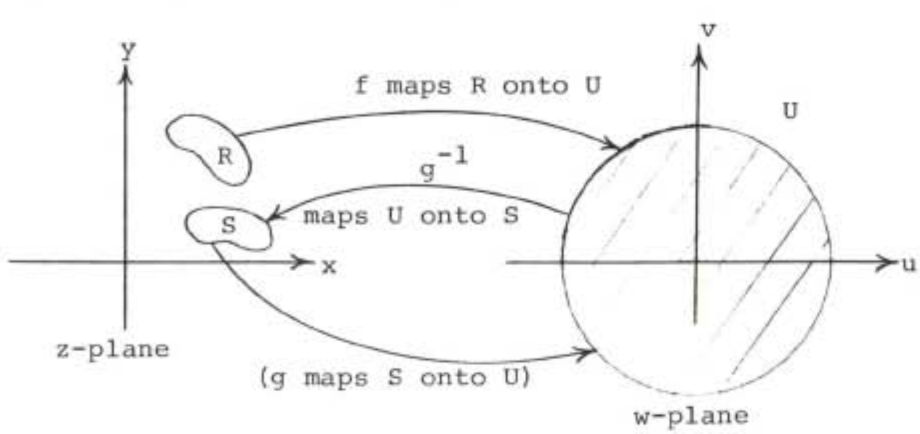
(1) We use the Riemann Mapping Theorem to find a function f which maps R conformally onto the unit disc $U = \{w: |w| < 1\}$.

(2) We find a function g which maps S conformally onto U.

(3) Since g (or f) is conformal, it is invertible. Hence, g^{-1} maps U onto S.

(4) Therefore, $g^{-1} \circ f$ is the desired conformal mapping of R onto S.

Pictorially,



1.6.8 (Optional)

a. Suppose f is defined by $f(z) = \frac{2z + i}{iz + 1}$ where $z \neq i$ [if $z = i$, $iz + 1 = 0$ so that f(z) is undefined]. Show that f is conformal in a sufficiently small neighborhood of each point z_o in the z-plane.

b. In particular, describe the behavior of f in a sufficiently small neighborhood of $z = 0$.

c. Do the same as in part (b) but now look at a small neighborhood of $z = 1$.

d. Describe the mapping f if f is defined by $f(z) = \frac{z - i}{iz + 1}$ $(z \neq i)$.

e. By computing $f'(z)$, show that if $f(z) = \frac{az + b}{cz + d}$ then f is conformal provided only that $ad - bc \neq 0$.

1.6.9 (Optional)

Let P_o be the point (0,0,1) on the sphere S defined by $x^2 + y^2 + z^2 = 1$ and let $P_1(x_1,y_1,z_1)$ be any other point on S. Find the point at which the line P_oP_1 intersects the xy-plane.

1.6.10 (Optional)

a. Suppose that f_1 and f_2 are any two members of the linear group. Show that $f_1 \circ f_2$ is also a member of the linear group (i.e., prove that the linear group is closed with respect to function composition).

b. Compute $f_1 \circ f_2$ and $f_2 \circ f_1$ if $f_1(z) = \frac{z + 1}{z - 1}$ and $f_2(z) = \frac{z}{z + 2}$.

c. Use long-division to show that if $c \neq 0$, $\frac{az + b}{cz + d}$ may be written in the form

$$\frac{bc - ad}{c^2}\left[\frac{1}{z + \frac{d}{c}}\right] + \frac{a}{c}.$$

1.6.11 (Optional)

a. Suppose $f(z) = \frac{az + b}{cz + d}$ where $ad - bc \neq 0$. Describe $f^{-1}(z)$.

b. Show that if f belongs to the linear group and $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$, then f is the identity function [i.e., $f(z) = z$ for every z].

c. Let z_1, z_2, and z_3 be any three distinct points in the z-plane. Show that there exists one and only one member f of the linear group such that $f(z_1) = 0$, $f(z_2) = 1$, and $f(z_3) = \infty$.

d. Let z_1, z_2, and z_3 be three arbitrarily chosen distinct points in the z-plane and let w_1, w_2, and w_3 be three arbitrarily chosen points in the w-plane. Show that there is exactly one member f of the linear group such that $f(z_1) = w_1$, $f(z_2) = w_2$, and $f(z_3) = w_3$.

e. Find the member of the linear group which maps 0 into 1, i into -1, and -i into 0.

f. The xy-plane is mapped conformally into the uv-plane by the linear mapping defined by; (0,0) is mapped onto (1,0); (0,1) is mapped onto (-1,0); and (0,-1) is mapped onto (0,0). Describe the mapping explicitly in the form

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}.$$

In particular, what is the image of the x-axis?