

Unit 5: The Cross Product

1. Lecture 1.050

**The Cross Product**

$\vec{A} \times \vec{B}$  is a vector such that

- $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin\theta$
- $\vec{A} \times \vec{B}$  is  $\perp$   $\vec{A}$  and  $\vec{B}$
- the sense is r.h. rule as  $\vec{A}$  is rotated into  $\vec{B}$  through the smaller angle

**Note:**

(1)  $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

(2)  $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$

$\vec{A} \times (\vec{B} \times \vec{C})$  is  $\perp$   $\vec{A}$  and  $\vec{B} \times \vec{C}$   
 $\vec{B} \times \vec{C}$  is  $\perp$   $\vec{B}$  and  $\vec{C}$   
 $\therefore \vec{A} \times (\vec{B} \times \vec{C})$  is parallel to the plane determined by  $\vec{B}$  and  $\vec{C}$   
 $(\vec{A} \times \vec{B}) \times \vec{C}$  is parallel to the plane determined by  $\vec{A}$  and  $\vec{B}$

a.

**Cross Product** has "a little" structure

$\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$

In Cartesian coordinates,

$\vec{A} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$   
 $\vec{B} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

Then:

$\vec{A} \times \vec{B} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$

**"Convenient" Memory Device**

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

b.

**Very Brief Look at Determinants**

Def:  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x_1 \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & y_3 \\ z_1 & z_3 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}$$

**Example 1**  
Find vector  $\perp$  to plane determined by  $A(1,2,3)$ ,  $B(5,9,4)$  and  $C(7,6,8)$

$\vec{AB} = 4\vec{i} + 7\vec{j} + \vec{k}$   
 $\vec{AC} = 6\vec{i} + 4\vec{j} + 5\vec{k}$

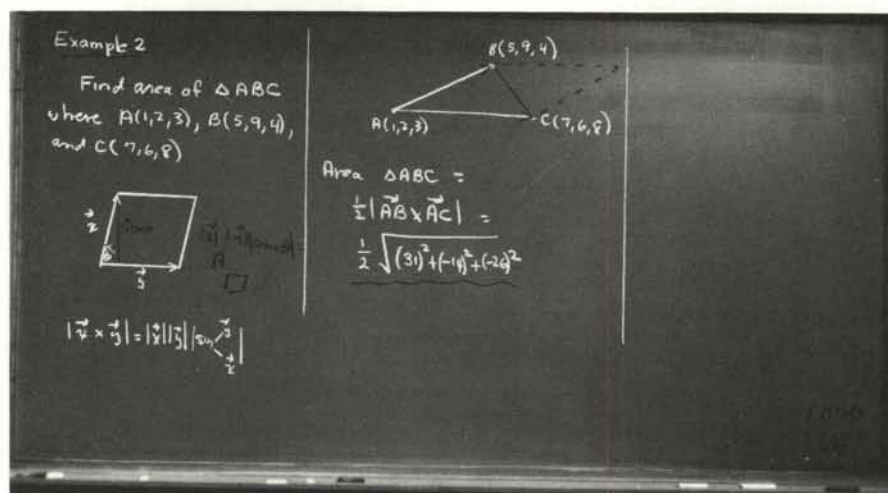
$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 7 & 1 \\ 6 & 4 & 5 \end{vmatrix}$

$$= \vec{i}(35-4) - \vec{j}(20-6) + \vec{k}(16-42)$$

$$= 31\vec{i} - 14\vec{j} - 26\vec{k}$$

c.

Lecture 1.050 continued



d.

Study Guide  
Block 1: Vector Arithmetic  
Unit 5: The Cross Product

---

2. Read Thomas, Sections 12.7 and 12.9.

3. Exercises:

1.5.1(L)

---

Find a vector which is perpendicular to both  $\vec{A} = 3\vec{i} + 4\vec{j} + 5\vec{k}$  and  $\vec{B} = 2\vec{i} + 6\vec{j} + 7\vec{k}$ .

1.5.2(L)

---

Let  $A(1,2,3)$ ,  $B(3,3,5)$  and  $C(4,8,1)$  be points in space.

- Find a vector perpendicular to the plane determined by  $A$ ,  $B$ , and  $C$ .
- Explain geometrically why  $\vec{AB} \times \vec{AC}$  and  $\vec{AB} \times \vec{BC}$  can differ, at most, only in their sense.
- With  $A$ ,  $B$ , and  $C$  as above, find the area of  $\triangle ABC$ .

1.5.3(L)

---

- Outline a method for finding the distance between two skew lines. (For the definition of a skew line, see the introduction to the solution of this exercise.)
- Translate the method in (a) into a form which utilizes concepts of vector arithmetic.
- Find the distance between the two skew lines one of which passes through the points  $A(1,2,3)$  and  $B(4,5,1)$  and the other of which passes through  $C(2,3,5)$  and  $D(3,6,8)$ .

1.5.4

---

Let  $A(2,3,4)$ ,  $B(5,6,8)$ ,  $C(4,5,9)$ , and  $D(6,11,14)$  be given points in space. Find the distance between the skew lines  $\vec{AB}$  and  $\vec{CD}$ .

1.5.5(L)

---

- Describe the direction of  $(\vec{A} \times \vec{B}) \times \vec{C}$ .
- Use the answer in (a) to show why  $(\vec{A} \times \vec{B}) \times \vec{C}$  and  $\vec{A} \times (\vec{B} \times \vec{C})$  need not be equal vectors.

1.5.6

---

Let  $\vec{A} = \vec{i} + 2\vec{j} + 3\vec{k}$ ,  $\vec{B} = 3\vec{i} + 5\vec{j} + 4\vec{k}$ , and  $\vec{C} = 6\vec{i} + 8\vec{j} + 9\vec{k}$ . Use the techniques of Exercise 1.5.5 to find a vector in the plane determined by  $\vec{A}$  and  $\vec{B}$ , and which is perpendicular to  $\vec{C}$ .

1.5.7

---

Use the result that  $(\vec{A} \times \vec{B}) \times \vec{C} = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{B} \cdot \vec{C})\vec{A}$ , together with the appropriate arithmetic properties of the cross product to express  $\vec{A} \times (\vec{B} \times \vec{C})$  as a linear combination of  $\vec{B}$  and  $\vec{C}$  (i.e., in the form  $p\vec{B} + q\vec{C}$ , where  $p$  and  $q$  are scalars).

1.5.8

---

- Express  $(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D})$  as a linear combination of  $\vec{A}$  and  $\vec{B}$ .
- Geometrically, what does the vector  $(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D})$  represent?

1.5.9

---

- Vectors are drawn from the origin to the points  $A$ ,  $B$ , and  $C$  in space. In terms of the vectors  $\vec{OA}$ ,  $\vec{OB}$ , and  $\vec{OC}$ , express the condition that these three vectors (or the four points  $O$ ,  $A$ ,  $B$ , and  $C$ ) lie in the same plane.
- Use the result of (a) to conclude that the vectors  $\vec{A}(1,1,1)$ ,  $\vec{B}(2,3,4)$ , and  $\vec{C}(3,4,5)$  originating at a common point lie in the same plane.
- A parallelepiped has one vertex at  $O(0,0,0)$  and three other vertices at  $A(1,1,1)$ ,  $B(2,4,3)$ , and  $C(3,4,5)$ . Find the volume of this parallelepiped.

MIT OpenCourseWare  
<http://ocw.mit.edu>

**Resource: Calculus Revisited: Multivariable Calculus**  
Prof. Herbert Gross

The following may not correspond to a particular course on MIT OpenCourseWare, but has been provided by the author as an individual learning resource.

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.