CALCULUS REVISITED
PART 2

A Self-Study Course

STUDY GUIDE

Block 1
Vector Arithmetic

Block 2
Vector Calculus

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INTRODUCTION

This self-study course consists of several elements which supplement one another. As in most courses, the central building block is the textbook. The remaining parts augment the text. First, there are the lectures. These are designed to give an overview of the material covered in the text and to supply motivation and insight in those areas where the oral word is more helpful than the written word.

Because the lectures are on film (or tape) it is assumed that you will be able to view a lecture more than once. You may use the lecture as an introductory overview and then review the unit by watching the lecture again when the rest of the assignment for that unit has been completed.

Yet, the fact remains that most students will not, for one reason or another, watch the lecture as often as might be advisable. For this reason, photographs of the blackboards, exactly as they appeared at the end of the lecture, have been made and reproduced as part of the study guide. Consequently, as you proceed through an assignment, there is always a rather convenient reminder and summary of the lecture. In fact, it might very well happen that once you have seen the lecture and done the assignment, the photographs of the blackboards will be sufficient to supply you with an "instant replay" whenever needed.

After the lecture, there are times when additional material is needed to bridge the gap between where the lecture ends and the text begins. Certain topics in the text are particularly difficult and as a result the student requires additional points of view or even a rehash of what's in the text. Frequently, a choice of approaches to a difficult topic is the best psychological boost to the student. Certain important topics are sometimes presented in the text in order to solve a specific problem, but it turns out that these topics have applications far beyond the specific problem in question; consequently the student would benefit from a more detailed explanation. Finally, there are certain topics that form a "twilight zone" for the student. Roughly speaking, these are the topics that the college professor assumes the student learned in high school and the high school teacher thinks he will learn in college. In such cases a student may need more explanation than is offered in the text. For these reasons, the course includes a volume entitled "Supplementary Notes."

Finally, it often happens that even under the assumptions that one has given the best possible lectures, has chosen the best possible textbook, and has written the best supplementary notes, there are still things the student does not learn unless he sees them occur in solutions to exercises. For this reason a major portion of the course consists of exercises together with in-depth solutions.

Some of the exercises serve as a vehicle (indeed, almost as an excuse) for introducing difficult but important topics in the guise of solutions to important exercises. When this occurs the exercise is labeled (L) to indicate to the student that it is a learning exercise, an exercise for which (even if he can solve the problem) he should read the solution because one or more important asides will be made there.

Other exercises are more routine and are supplied simply so that the student can test whether he can handle the material. These exercises appear without special designation.

The final type of exercise is called the optional exercise. It often happens that in the learning of a new concept one has to see beyond where he is in order to appreciate better his present position. For example, it often happens that a student does not begin to master algebra, geometry, or trigonometry until he studies calculus where these topics are used as a means toward an end rather than as an end in themselves, or he learns calculus better while he is studying differential equations, etc. At any rate, for this reason we have interjected optional problems for the purpose of giving new ideas additional exposure. An optional exercise together with its solution forms a step-by-step supplement to the material already presented. Whenever an optional problem is given, there is an explanation as to why it is assigned and what it hopes to accomplish.

Actually, it is the hope of the author that the student will not distinguish between optional and "regular" exercises (to an author nothing is optional and everything is important), but we recognize that certain students may not have either the desire or the time to delve in depth into certain topics. Accordingly, wherever we feel that there is no serious damage done by the omission of the exercise, we have called it optional.

Two other features round out our course. Since many people who will be taking this course may have studied some of the topics previously, we have included a pretest at the beginning of each new block of material. The nature of this pretest is to let the student know whether he needs the material contained in the block. That is, if he can pass the pretest comfortably, he may elect to by-pass the block. It may be of interest to know that each pretest problem occurs as a learning exercise within the block, so that the student who has trouble with such an exercise in the pretest can rest assured he will see a detailed solution later.

As important as the pretest is the post-test or what is more colloquially known as the quiz. Somehow or other, there is no substitute for a comprehensive test to see what the student has retained. For this reason there is a "final examination" at the end of each block. The correct answers together with rather detailed solutions are supplied so that the student can better analyze his difficulties.

In summary, then, our typical format for a lesson unit is:

1. See a lecture.
2. Read some supplementary notes.
3. Read a portion of the text.
4. Do the exercises.

When assigned, these four steps almost always occur in the given order, but there are some assignments in which (1) and/or (2) are omitted, and there are a few places where the supplementary notes form the only reading assignment, especially in the treatment of topics not covered in the text.

Finally, I would like to acknowledge the very able assistance I have received from several people in the preparation of this self-study course. First and foremost, I am deeply indebted to John T. Fitch, the manager of our self-study development program. He discussed and helped me plan the lectures and written material -- unit by unit. He made suggestions, offered improvements, and, in many cases, put himself in the role of the student to help me "tone down" certain topics to the extent that they became (hopefully) more understandable to the student. In addition to all this, he was a friend and colleague, and this went a long way towards making a very difficult undertaking more palatable for me.

I would also like to thank Harold S. Mickley, the first director of CAES, whose idea it was to make "Calculus Revisited" available as a self-study course. Most of this present course reflects his ideas as to what constitutes a meaningful continuing education, calculus course. He, too, during his stay at the Center was a constant source of inspiration to me, and, in a certain sense, this course belongs more to him than to anyone else.

If you think that having to read all this material is difficult, imagine what it would have been like to have had to type the entire manuscript. Yet this job was accomplished, in an efficient and good-natured manner, by our able staff of secretaries - in addition to maintaining all the other responsibilities of their office. In particular, I am grateful to Miss Elise Pelletier who worked on the manuscript from its very inception and to Mrs. Richard Borken for their help in the preparation of the manuscript.

I hope that your studying this course will be as rewarding and enjoyable as preparing the course has been to me. Good luck.

Cambridge, Massachusetts
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Herbert I. Gross

Study Guide

BLOCK 1:

VECTOR ARITHMETIC

Pretest

1. Explain why the following argument is invalid.

   Some A's are not B's.
   Some B's are not C's.
   Therefore, some A's are not C's.

2. Let A(1,1), B(4,5), and C(6,13) be points in the xy-plane. Find the equation of the line which bisects $\measuredangle$ BAC.

3. Given the points A(1,2,3), B(2,4,1), and C(4,8,5), determine the point at which the medians of $\triangle$ ABC intersect.

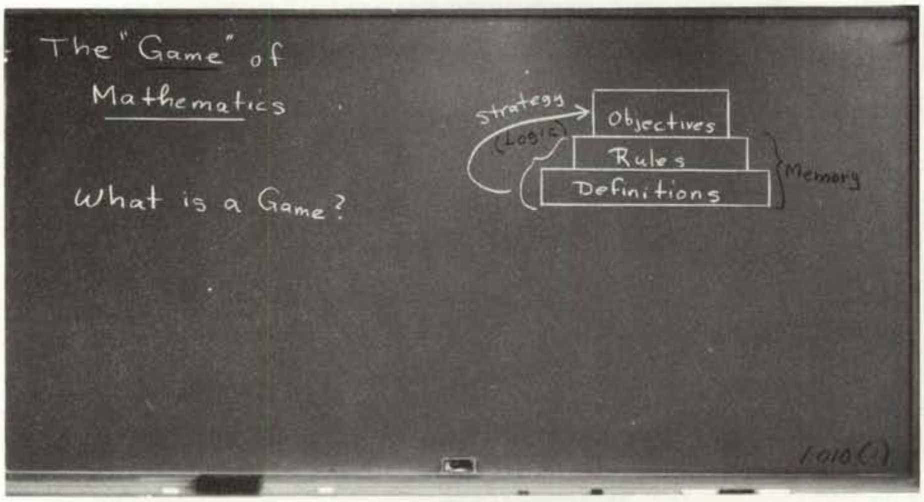
4. Let $\vec{A} = \vec{i} + \vec{j} + \vec{k}$, $\vec{B} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, and $\vec{C} = x\vec{i} + y\vec{j} + z\vec{k}$. How are x, y, and z related if $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$?

5. Given the points A(1,2,3), B(3,3,5), and C(4,8,1,), find a vector perpendicular to the plane determined by A, B, and C.

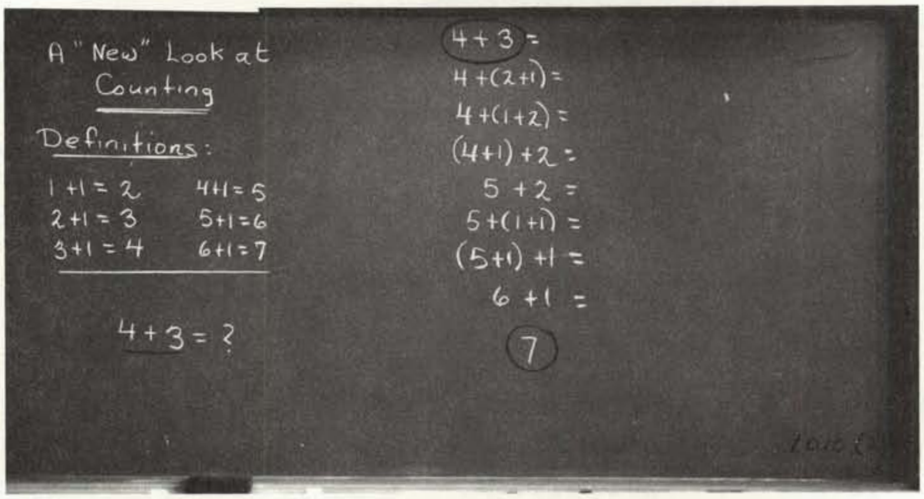
6. Find the distance from the point (7,8,9) to the plane whose equation is $2x + 3y + 6z = 8$.

Unit 1: An Introduction to Mathematical Structure

1. Lecture 1.010



a.



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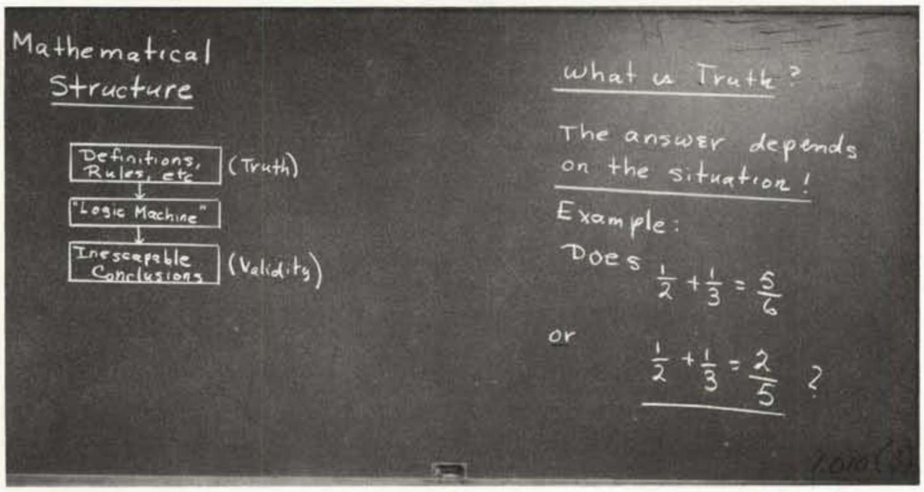
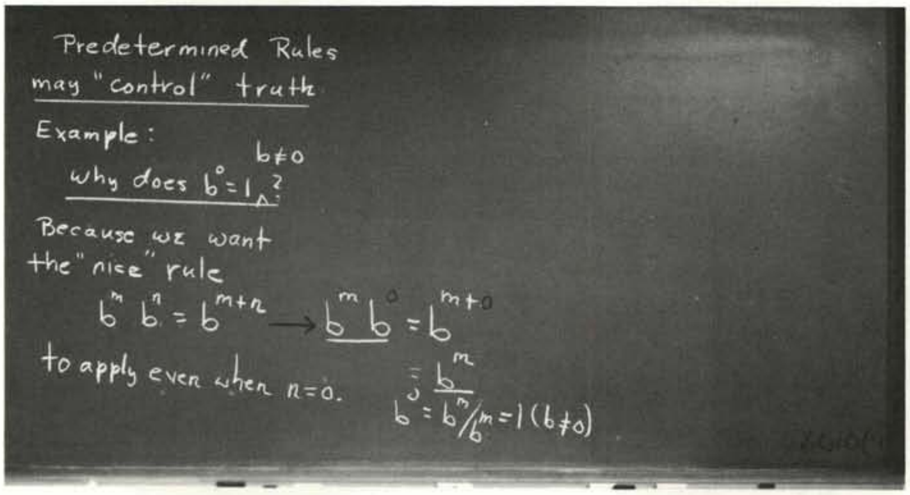


c.

Lecture 1.010 continued

d. 

2. Read Supplementary Notes, Chapter 1.

3. Exercises:

1.1.1(L)

a. Discuss the validity of the argument:

No A's are B's
No B's are C's
Therefore, no A's are C's

b. Discuss the validity of the argument that if $A \cap B = \emptyset$ and if $B \cap C = \emptyset$ then $A \cap C = \emptyset$.

1.1.2

a. Discuss the validity of the argument that if some A's are not B's and if some B's are not C's then some A's are not C's.

b. Discuss the validity of the argument that if $A \cap B' \neq \emptyset$ and if $B \cap C' \neq \emptyset$ then $A \cap C' \neq \emptyset$.

c. Discuss the validity of the argument and the truth of the conclusion in the following:

Some numbers which are divisible by 2 are not divisible by 7; and some numbers which are divisible by 7 are not divisible by 3. Therefore, some numbers which are divisible by 2 are not divisible by 3.

1.1.3

a. Do the statements "Some A's are not B's" and "Some B's are not A's" have the same meaning? Why?

b. Repeat part (a) in terms of sets.

1.1.4

a. Define the relation R by aRb means that a lives next door to b. Is R reflexive? Transitive? Symmetric?

(continued on next page)

1.1.4 continued

b. Define R by aRb means that $2a + b = 15$. Test the truth of the statements 7R1, 1R13, and 7R13; then decide whether R is transitive.

c. With R as in part (b) test the truth of 1R7. Is R symmetric? Explain.

d. Define R by aRb means $a + b = 15$. Is R symmetric? Explain.

1.1.5(L)

a. Determine whether the following argument is valid and then comment on the truth of the conclusion.

Giving handouts to beggars helps the poor
Helping the poor is good for society
Giving handouts to beggars is good for society

b. Using the fact that $(n + 1)! = (n + 1)n!$, explain why $0! = 1$. Could 0! have been defined differently? Explain your answer to the last question in terms of truth versus validity.

1.1.6(L)

a. Mimic our development of rules A-1 through A-5 in the supplementary notes to "invent" analogous rules M-1 through M-5 for multiplication.

b. The distributive rule for arithmetic says that for any numbers a, b, and c, $a(b + c) = ab + ac$. By computing $a(0 + 0)$ in two different ways, prove that $a0 = 0$ for every number, a.

c. Utilize the parallel structure of multiplication to addition and prove that if $a \neq 0$ and $ab = ac$, then $b = c$.

1.1.7

Utilize the results of parts (b) and (c) of Exercise 1.1.6 to prove that if $a \neq 0$ and $ab = 0$ then $b = 0$.

Comment

The following two exercises are optional. They involve the concept of modular arithmetic. Our reason for studying it is that it serves as a completely different number system which still obeys our rules E-1 through E-5, A-1 through A-5, and M-1 through M-5, with the possible exception of M-5.

Thus, we have a way of seeing how parallel structure is utilized in the study of modular arithmetic once we have studied ordinary arithmetic. In addition to reinforcing the ideas of this unit, the exercises serve as a link between the numerical arithmetic studied in this unit and the vector arithmetic that will be studied in the next unit.

The point is that while the very mature student can probably go directly from numerical arithmetic to vector arithmetic, many students can benefit from the transitional treatment given in Exercises 1.1.8(L) and 1.1.9.

1.1.8(L)

Define "=" in a new way as follows. For any numbers a and b we will write a = b if and only if a and b leave the same remainder when divided by 7.

a. Show that "=" as defined above is an equivalence relation.

b. In terms of our definition, 4 + 5 = 2 since both 9 and 2 leave the same remainder when divided by 7. Proceed in this way to write down the addition and multiplication tables for the numbers 0, 1, 2, 3, 4, 5, and 6.

c. Use your tables constructed in (b) to compute:

(1) whether $4 + (5 + 6) = (4 + 5) + 6$
(2) whether $(3 + 4) \times 5 = 3 + (4 \times 5)$
(3) the value of 3^{105}

d. If 3^{-1} is the number which must be multiplied by 3 to yield 1, find the value of 3^{-1} from the tables in (b).

e. Use the tables in (b) to find the value of 1^{-1}, 2^{-1}, 4^{-1}, 5^{-1}, and 6^{-1}.

(continued on next page)

1.1.8(L) continued

f. In this new system of arithmetic, if $a \neq 0$ and $ab = 0$, must $b = 0$? Explain.

1.1.9

a. Make the arithmetic tables for modular-6 arithmetic.

b. From your tables,

(1) compare $(3 \times 5) \times 2$ and $3 \times (5 \times 2)$

(2) compare $(3 \times 5) + 4$ and $3 \times (5 + 4)$

(3) compute the value of 5^{1000}.

c. Discuss the value of

(1) 5^{-1}

(2) 3^{-1}

d. In this system if $ab = 0$ and $a \neq 0$, must $b = 0$? Explain.

Resource: Calculus Revisited: Multivariable Calculus
Prof. Herbert Gross