BLOCK 4:
MATRIX ALGEBRA

Pretest

1. $X = \begin{bmatrix} 89 & 73 \\ -50 & -41 \end{bmatrix}$

2. $A^{-1} = \begin{bmatrix} 4 & -3 & 1 \\ -\frac{13}{8} & 1 & -\frac{1}{8} \\ \frac{3}{8} & 0 & -\frac{1}{8} \end{bmatrix}$

3. $\left.\begin{array}{l} b_3 = 5b_1 - b_2 \\ b_4 = 7b_1 - 2b_2 \end{array}\right\}$

4. $(x,y) \approx (3.00008, 1.99999)$

5. $\dfrac{dx}{dz} = \dfrac{-(y - z)^2}{x^2 + y^2 + z^2 + 1}$

6. The maximum value is 1 and occurs when $x = 1$, $y = 0$, $z = 0$.
The minimum value is $\frac{5}{9}$ and occurs when $x = \frac{1}{3}$, $y = \frac{2}{3}$, $z = 0$.

Unit 1: Linear Equations and Introduction to Matrices

4.1.1

To compute the product of two matrices, we find the term in the i^{th} row, j^{th} column by "dotting" the i^{th} row of the first matrix with the j^{th} column of the second matrix. (This is why the number of columns in the first matrix must equal the number of rows in the second matrix).

Thus, the term in the 2^{nd} row, 3^{rd} column of

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 6 & 2 & 7 \\ 4 & 5 & 1 & 8 \\ 4 & 7 & 9 & 5 \end{pmatrix} \tag{1}$$

is given by $(3,1,2)\cdot(2,1,9)$ or in terms of the form given in (1)

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 6 & 2 & 7 \\ 4 & 5 & 1 & 8 \\ 4 & 7 & 9 & 5 \end{pmatrix}$$

Continuing in this way, we obtain

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 6 & 2 & 7 \\ 4 & 5 & 1 & 8 \\ 4 & 7 & 9 & 5 \end{pmatrix} =$$

$$= \begin{bmatrix} (1,2,3)\cdot(3,4,4) & (1,2,3)\cdot(6,5,7) & (1,2,3)\cdot(2,1,9) & (1,2,3)\cdot(7,8,5) \\ (3,1,2)\cdot(3,4,4) & (3,1,2)\cdot(6,5,7) & (3,1,2)\cdot(2,1,9) & (3,1,2)\cdot(7,8,5) \end{bmatrix}$$

$$= \begin{bmatrix} 3+8+12 & 6+10+21 & 2+2+27 & 7+16+15 \\ 9+4+8 & 18+5+14 & 6+1+18 & 21+8+10 \end{bmatrix}$$

$$= \begin{pmatrix} 23 & 37 & 31 & 38 \\ 21 & 37 & 25 & 39 \end{pmatrix} \tag{2}$$

4.1.1 continued

If we now recall that our definition of matrix multiplication was motivated by the chain rule for systems of linear equations, equation (2) tells us at once that if

$$\left.\begin{aligned} z_1 &= y_1 + 2y_2 + 3y_3 \\ z_2 &= 3y_1 + y_2 + 2y_3 \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} y_1 &= 3x_1 + 6x_2 + 2x_3 + 7x_4 \\ y_2 &= 4x_1 + 5x_2 + x_3 + 8x_4 \\ y_3 &= 4x_1 + 7x_2 + 9x_3 + 5x_4 \end{aligned}\right\} \tag{4}$$

then

$$z_1 = 23x_1 + 37x_2 + 31x_3 + 38x_4$$

$$z_2 = 21x_1 + 37x_2 + 25x_3 + 39x_4$$

This result may, of course, be obtained directly by replacing y_1, y_2, and y_3 in (3) by their values in (4). In fact, this is how we arrived at the recipe for matrix products.

4.1.2

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 2 \\ 5 & 7 & 9 \end{pmatrix} =$$

$$= \begin{bmatrix} (1,1,2)\cdot(3,2,5) & (1,1,2)\cdot(4,1,7) & (1,1,2)\cdot(5,2,9) \\ (2,3,2)\cdot(3,2,5) & (2,3,2)\cdot(4,1,7) & (2,3,2)\cdot(5,2,9) \\ (3,4,5)\cdot(3,2,5) & (3,4,5)\cdot(4,1,7) & (3,4,5)\cdot(5,2,9) \end{bmatrix}$$

4.1.2 continued

$$= \begin{bmatrix} 3+2+10 & 4+1+14 & 5+2+18 \\ 6+6+10 & 8+3+14 & 10+6+18 \\ 9+8+25 & 12+4+35 & 15+8+45 \end{bmatrix}$$

$$= \begin{pmatrix} 15 & 19 & 25 \\ 22 & 25 & 34 \\ 42 & 51 & 68 \end{pmatrix} \qquad (1)$$

$$\begin{pmatrix} 3 & 4 & 5 \\ 2 & 1 & 2 \\ 5 & 7 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} =$$

$$= \begin{bmatrix} (3,4,5)\cdot(1,2,3) & (3,4,5)\cdot(1,3,4) & (3,4,5)\cdot(2,2,5) \\ (2,1,2)\cdot(1,2,3) & (2,1,2)\cdot(1,3,4) & (2,1,2)\cdot(2,2,5) \\ (5,7,9)\cdot(1,2,3) & (5,7,9)\cdot(1,3,4) & (5,7,9)\cdot(2,2,5) \end{bmatrix}$$

$$= \begin{bmatrix} 3+8+15 & 3+12+20 & 6+8+25 \\ 2+2+6 & 2+3+8 & 2+2+10 \\ 5+14+27 & 5+21+36 & 10+14+45 \end{bmatrix}$$

$$= \begin{pmatrix} 26 & 35 & 39 \\ 10 & 13 & 14 \\ 46 & 62 & 69 \end{pmatrix} \qquad (2)$$

Equation (1) tells us that

$$\left.\begin{aligned} z_1 &= y_1 + y_2 + 2y_3 \\ z_2 &= 2y_1 + 3y_2 + 2y_3 \\ z_3 &= 3y_1 + 4y_2 + 5y_3 \end{aligned}\right\} \qquad (3)$$

and

$$\left.\begin{aligned} y_1 &= 3x_1 + 4x_2 + 5x_3 \\ y_2 &= 2x_1 + x_2 + 2x_3 \\ y_3 &= 5x_1 + 7x_2 + 9x_3 \end{aligned}\right\} \qquad (4)$$

4.1.2 continued

then

$$\left.\begin{aligned} z_1 &= 15x_1 + 19x_2 + 25x_3 \\ z_2 &= 22x_1 + 25x_2 + 34x_3 \\ z_3 &= 42x_1 + 51x_2 + 68x_3 \end{aligned}\right\} \qquad (5)$$

Equation (2) tells us that if

$$\left.\begin{aligned} z_1 &= 3y_1 + 4y_2 + 5y_3 \\ z_2 &= 2y_1 + y_2 + 2y_3 \\ z_3 &= 5y_1 + 7y_2 + 9y_3 \end{aligned}\right\} \qquad (6)$$

and

$$\left.\begin{aligned} y_1 &= x_1 + x_2 + 2x_3 \\ y_2 &= 2x_1 + 3x_2 + 2x_3 \\ y_3 &= 3x_1 + 4x_2 + 5x_3 \end{aligned}\right\} \qquad (7)$$

then

$$\left.\begin{aligned} z_1 &= 26x_1 + 35x_2 + 39x_3 \\ z_2 &= 10x_1 + 13x_2 + 14x_3 \\ z_3 &= 46x_1 + 62x_2 + 69x_3 \end{aligned}\right\} \qquad (8)$$

Note:

The first part of this exercise shows us that matrix multiplication is not commutative. That is, the product of two matrices depends on the order in which the matrices are written. (Of course, we already knew this in theory since we saw that we could multiply an

4.1.2 continued

m x k matrix by a k x n matrix, but if we interchange the order of the factors we are then multiplying a k x n matrix by an m x k matrix, and by definition this makes no sense if $n \neq m$. The point is that when we multiply two n x n matrices, the product makes sense regardless of the order of the two matrices, but the product may change if we change the order of the matrices.)

The second part of this exercise shows us why the result of the first part should not be surprising. Namely, interchanging the matrices is equivalent to interchanging the coefficients of the y's with those of the x's. For example, in this exercise, compare equations (3) and (4) with equations (6) and (7). In general, such an interchanging of coefficients will affect how the z's look in terms of the x's. A comparison of equations (5) and (8) shows how we do get different relationships in this particular exercise.

In terms of our "game" of mathematics, we observe that once our definitions and rules are accepted based on what we call reality, we must let the "chips fall where they will"; and the best we can do is show in terms of our model why certain "non-intuitive" results are actually quite natural.

4.1.3

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{bmatrix} (1,1,2)\cdot(1,0,0) & (1,1,2)\cdot(0,1,0) & (1,1,2)\cdot(0,0,1) \\ (2,3,2)\cdot(1,0,0) & (2,3,2)\cdot(0,1,0) & (2,3,2)\cdot(0,0,1) \\ (3,4,5)\cdot(1,0,0) & (3,4,5)\cdot(0,1,0) & (3,4,5)\cdot(0,0,1) \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} . \tag{1}$$

Comparing both sides of equation (1), we notice that multiplying

4.1.3 continued

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \text{ by } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ yields } \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \text{ again.}$$

A closer examination of how equation (1) was obtained seems to indicate that this result hinged more on the structure of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

than on the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix}.$$

To check this out, let us replace

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \text{ by } \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

and look at

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We then see that

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{bmatrix} (a,b,c)\cdot(1,0,0) & (a,b,c)\cdot(0,1,0) & (a,b,c)\cdot(0,0,1) \\ (d,e,f)\cdot(1,0,0) & (d,e,f)\cdot(0,1,0) & (d,e,f)\cdot(0,0,1) \\ (g,h,i)\cdot(1,0,0) & (g,h,i)\cdot(0,1,0) & (g,h,i)\cdot(0,0,1) \end{bmatrix}$$

4.1.3 continued

$$= \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} . \tag{2}$$

Since

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

denotes <u>any</u> 3 x 3 matrix, we see from (2) that the result of multiplying any 3 x 3 matrix <u>on the right</u>* by the 3 x 3 matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the original matrix.

As for the remaining part of this exercise, we also observe that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} =$$

$$= \begin{bmatrix} (1,0,0)\cdot(a,d,g) & (1,0,0)\cdot(b,e,h) & (1,0,0)\cdot(c,f,i) \\ (0,1,0)\cdot(a,d,g) & (0,1,0)\cdot(b,e,h) & (0,1,0)\cdot(c,f,i) \\ (0,0,1)\cdot(a,d,g) & (0,0,1)\cdot(b,e,h) & (0,0,1)\cdot(c,f,i) \end{bmatrix}$$

$$= \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} . \tag{3}$$

Thus, a comparison of equations (2) and (3) shows us that if

*Since we have seen that matrix multiplication need not be commutative, it is important to fix the order of the factors when we talk about a matrix product.

4.1.3 continued

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is any 3 x 3 matrix, then

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad (4)$$

(Obviously, there is nothing special about 3 x 3 matrices. That is, similar results would apply to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if we were dealing with 2 x 2 matrices,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

if we were dealing the 4 x 4 matrices, etc.)

Equation (4) tells us also that the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>commutes</u> with every 3 x 3 matrix.

Finally, to see what equations (2) and (3) mean with respect to systems of linear equations, we have that if

$$z_1 = ay_1 + by_2 + cy_3$$

$$z_2 = dy_1 + ey_2 + fy_3$$

$$z_3 = gy_1 + hy_2 + iy_3$$

4.1.3 continued

and

$$\left.\begin{aligned} y_1 &= 1x_1 + 0x_2 + 0x_3 \\ y_2 &= 0x_1 + 1x_2 + 0x_3 \\ y_3 &= 0x_1 + 0x_2 + 1x_3 \end{aligned}\right\} \quad \text{i.e.,} \quad \left\{\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ y_3 &= x_3 \end{aligned}\right.$$

then

$$\begin{aligned} z_1 &= ax_1 + bx_2 + cx_3 \\ z_2 &= dx_1 + ex_2 + fx_3 \\ z_3 &= gx_1 + hx_2 + ix_3 \end{aligned}$$

Similarly, if

$$\left.\begin{aligned} z_1 &= y_1 \\ z_2 &= y_2 \\ z_3 &= y_3 \end{aligned}\right\} \quad \text{i.e.,} \quad \left\{\begin{aligned} z_1 &= 1y_1 + 0y_2 + 0y_3 \\ z_2 &= 0y_1 + 1y_2 + 0y_3 \\ z_3 &= 0y_1 + 0y_2 + 1y_3 \end{aligned}\right.$$

and

$$\begin{aligned} y_1 &= ax_1 + bx_2 + cx_3 \\ y_2 &= dx_1 + ex_2 + fx_3 \\ y_3 &= gx_1 + hx_2 + ix_3 \end{aligned}$$

then

$$\begin{aligned} z_1 &= ax_1 + bx_2 + cx_3 \\ z_2 &= dx_1 + ex_2 + fx_3 \\ z_3 &= gx_1 + hx_2 + ix_3 \end{aligned}$$

4.1.4

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} 7 & 3 & -4 \\ -4 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix} =$$

$$= \begin{bmatrix} (1,1,2)\cdot(7,-4,-1) & (1,1,2)\cdot(3,-1,-1) & (1,1,2)\cdot(-4,2,1) \\ (2,3,2)\cdot(7,-4,-1) & (2,3,2)\cdot(3,-1,-1) & (2,3,2)\cdot(-4,2,1) \\ (3,4,5)\cdot(7,-4,-1) & (3,4,5)\cdot(3,-1,-1) & (3,4,5)\cdot(-4,2,1) \end{bmatrix}$$

$$= \begin{bmatrix} 7 - 4 - 2 & 3 - 1 - 2 & -4 + 2 + 2 \\ 14 - 12 - 2 & 6 - 3 - 2 & -8 + 6 + 2 \\ 21 - 16 - 5 & 9 - 4 - 5 & -12 + 8 + 5 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 7 & 3 & -4 \\ -4 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} =$$

$$= \begin{bmatrix} (7,3,-4)\cdot(1,2,3) & (7,3,-4)\cdot(1,3,4) & (7,3,-4)\cdot(2,2,5) \\ (-4,-1,2)\cdot(1,2,3) & (-4,-1,2)\cdot(1,3,4) & (-4,-1,2)\cdot(2,2,5) \\ (-1,-1,1)\cdot(1,2,3) & (-1,-1,1)\cdot(1,3,4) & (-1,-1,1)\cdot(2,2,5) \end{bmatrix}$$

$$= \begin{bmatrix} 7 + 6 - 12 & 7 + 9 - 16 & 14 + 6 - 20 \\ -4 - 2 + 6 & -4 - 3 + 8 & -8 - 2 + 10 \\ -1 - 2 + 3 & -1 - 3 + 4 & -2 - 2 + 5 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2}$$

A comparison of (1) and (2) shows us, among other things, that the matrices

4.1.4 continued

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 7 & 3 & -4 \\ -4 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

commute with respect to matrix multiplication.

In terms of systems of linear equations, equation (1) tells us that if

$$\left.\begin{aligned} z_1 &= y_1 + y_2 + 2y_3 \\ z_2 &= 2y_1 + 3y_2 + 2y_3 \\ z_3 &= 3y_1 + 4y_2 + 5y_3 \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} y_1 &= 7x_1 + 3x_2 - 4x_3 \\ y_2 &= -4x_1 - x_2 + 2x_3 \\ y_3 &= -x_1 - x_2 + x_3 \end{aligned}\right\} \tag{4}$$

then

$$\left.\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= x_3 \end{aligned}\right\} \tag{5}$$

This, in turn, says that the system of equations (4) "undoes" the system of equations (3), or that (4) is the inverse of (3). More specifically, if we were to take the system (3) and solve explicitly for y_1, y_2, and y_3 in terms of z_1, z_2, and z_3 (and the details of this are left to a later unit although the interested reader may solve this system himself), we would obtain

4.1.4 continued

$$\left.\begin{aligned} y_1 &= 7z_1 + 3x_2 - 4z_3 \\ y_2 &= -4z_1 - z_2 - 2z_3 \\ y_3 &= -z_1 - z_2 + z_3 \end{aligned}\right\} \qquad (6)$$

Notice that if we replace x_1, x_2, and x_3 in (4) by their values in (5), we do obtain equation (6).

A similar discussion applies to the interpretation of equations (2) in terms of systems of linear equations.

As a final note, let us observe that while it is true that matrix multiplication is not commutative, this does not mean that there aren't many examples of matrix multiplication in which the multiplication is commutative. What is important is that we be careful not to interchange the order of factors in a matrix product since, except for certain special cases, the product will depend on the order of the factors.

4.1.5

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix} =$$

$$= \begin{bmatrix} (2,0,0)\cdot(5,0,0) & (2,0,0)\cdot(0,6,0) & (2,0,0)\cdot(0,0,7) \\ (0,3,0)\cdot(5,0,0) & (0,3,0)\cdot(0,6,0) & (0,3,0)\cdot(0,0,7) \\ (0,0,4)\cdot(5,0,0) & (0,0,4)\cdot(0,6,0) & (0,0,4)\cdot(0,0,7) \end{bmatrix}$$

$$= \begin{pmatrix} 10 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 28 \end{pmatrix} \qquad (1)$$

An n x n matrix is called a <u>diagonal matrix</u> if each entry not on the main diagonal (i.e., each entry a_{ij} for which $i \neq j$) is zero. Thus, equation (1) seems to indicate that the product of two diagonal matrices is also a diagonal matrix and that the product is obtained by multiplying the diagonal entries term by term. More generally

4.1.5 continued

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}\begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix} =$$

$$= \begin{bmatrix} (a_1, 0, 0)\cdot(b_1,0,0) & (a_1, 0, 0)\cdot(0,b_2,0) & (a_1, 0, 0)\cdot(0,0,b_3) \\ (0,a_2, 0)\cdot(b_1,0,0) & (0,a_2, 0)\cdot(0,b_2,0) & (0,a_2, 0)\cdot(0,0,b_3) \\ (0, 0,a_3)\cdot(b_1,0,0) & (0, 0,a_3)\cdot(0,b_2,0) & (0, 0,a_3)\cdot(0,0,b_3) \end{bmatrix}$$

$$= \begin{pmatrix} a_1b_1 & 0 & 0 \\ 0 & a_2b_2 & 0 \\ 0 & 0 & a_3b_3 \end{pmatrix} \qquad (2)$$

Notice also that since $a_1b_1 = b_1a_1$, etc., it follows that the product of two diagonal matrices does not depend on the order of the factors. That is, with respect to matrix multiplication, diagonal matrices commute.

Finally, it is easy to interpret the result about diagonal matrices in terms of systems of linear equations. For example, equation (2) tells us that if

$$\left.\begin{aligned} z_1 &= a_1y_1 \\ z_2 &= a_2y_2 \\ z_3 &= a_3y_3 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} y_1 &= b_1x_1 \\ y_2 &= b_2x_2 \\ y_3 &= b_3x_3 \end{aligned}\right\}$$

then

4.1.5 continued

$$\left.\begin{aligned} z_1 &= a_1b_1x_1 \\ z_2 &= a_2b_2x_2 \\ z_3 &= a_3b_3x_3 \end{aligned}\right\}$$

4.1.6

a. $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} =$$

$$= \begin{bmatrix} (1,0,1)\cdot(a_1,b_1,c_1) & (1,0,1)\cdot(a_2,b_2,c_2) & (1,0,1)\cdot(a_3,b_3,c_3) \\ (0,1,0)\cdot(a_1,b_1,c_1) & (0,1,0)\cdot(a_2,b_2,c_2) & (0,1,0)\cdot(a_3,b_3,c_3) \\ (0,0,1)\cdot(a_1,b_1,c_1) & (0,0,1)\cdot(a_2,b_2,c_2) & (0,0,1)\cdot(a_3,b_3,c_3) \end{bmatrix}$$

$$= \begin{bmatrix} a_1+c_1 & a_2+c_2 & a_3+c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \tag{1}$$

Equation (1) shows us that the result of multiplying

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

on the left by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is equivalent to adding, term by term, the third row of the matrix

4.1.6 continued

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

to the first row, and leaving the other rows as is.

b. If we let E_{ij} $(i \neq j)$ denote the n x n matrix each of whose elements on the main diagonal and in the i^{th} row j^{th} column is 1 and everywhere else 0, then $E_{ij}A$ simply replaces the i^{th} row of A by the term by term sum of the i^{th} row plus the j^{th} row. In our particular example, we had $E_{13}A$ where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

and the product was the matrix A with the 1st row replaced by the term by term sum of the first and third rows.

To prove this result in general it is best to work abstractly. Let e_{rk} denote the entry in the rth row, kth column of E_{ij} and let a_{rk} denote the corresponding element of A where E_{ij} and A are now n x n matrices. Now by the definition of E_{ij}, $e_{rk} = 0$ <u>except</u> when $r = k$ <u>or</u> $r = i$ and $k = j$, in which cases $e_{rk} = 1$.

If we now look at the entry in the rth row, sth column of $E_{ij}A$, we know that this entry is obtained by dotting the rth row of E_{ij} with the sth column of A. That is, in sigma-notation, the entry in the rth row, sth column of $E_{ij}A$ is given by

$$\sum_{k=1}^{n} e_{rk}a_{ks}. \tag{2}$$

If $r \neq i$ then $e_{rk} = 0$ except when $k = r$ in which case $e_{rk} = e_{rr} = 1$, so that the sum in (2) reduces to the single term

$$e_{rr}a_{rs} = a_{rs} \tag{3}$$

4.1.6 continued

(since $e_{rk} = 0$ otherwise).

Combining (3) with (2), we have that the term in the rth row, sth column of $E_{ij}A$ is a_{rs} if $r \neq i$. But, a_{rs} is the term in the rth row, sth column of A, and this demonstrates that the entry of $E_{ij}A$ in the rth row, sth column is the same as that for A provided $r \neq i$.

If $r = i$ then e_{rk} is equal to 1 for <u>two</u> different values of k; namely, $k = r$ and $k = j$ (since except for the diagonal terms only e_{ij} is different from 0 and in fact is equal to 1 by definition). At any rate, in this case, the sum in (2) reduces to the two terms

$$e_{rr}a_{rs} + e_{rj}a_{js} =$$

$$e_{ii}a_{is} + e_{ij}a_{js} =$$

$$a_{is} + a_{js} \tag{4}$$

From (4), we see that the term in the ith row, sth column of $E_{ij}A$ is obtained by adding the term in the ith row, sth column of A to the term in the jth row, sth column of s; and this is the same as saying that the ith row of $E_{ij}A$ is obtained by adding the ith row of A, <u>term by term</u>, to the jth row of A.

To see this idea more concretely, let us look at $E_{23}A$ where E_{23} and A are both 4 x 4 matrices. E_{23} has 1's along the main diagonal and 0's everywhere else except in the 2nd row, 3rd column where the entry is also 1. Thus,

$$E_{23}A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

If we look at any row of E_{23} <u>except the 2nd</u>, we see that multiplying A on the left by E_{23} doesn't change A. (Compare with the result of Exercise 4.1.3.) For example, the first row of $E_{23}A$ is given by

4.1.6 continued

$$\begin{bmatrix}(1,0,0,0)\cdot(a_{11},a_{21},a_{31},a_{41}) & (1,0,0,0)\cdot(a_{12},a_{22},a_{32},a_{42}) & (1,0,0,0)\cdot(a_{13},a_{23},a_{33},a_{43}) & (1,0,0,0)\cdot(a_{14},a_{24},a_{34},a_{44})\end{bmatrix}$$

or

$$\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\end{bmatrix}.$$

When we multiply the 2nd row of E_{23} by A, we obtain

$$\begin{bmatrix}(0,1,1,0)\cdot(a_{11},a_{21},a_{31},a_{41}) & (0,1,1,0)\cdot(a_{12},a_{22},a_{32},a_{42}) & (0,1,1,0)\cdot(a_{13},a_{23},a_{33},a_{43}) & (0,1,1,0)\cdot(a_{14},a_{24},a_{34},a_{44})\end{bmatrix} =$$

$$= \begin{bmatrix}a_{21}+a_{31} & a_{22}+a_{32} & a_{23}+a_{33} & a_{24}+a_{34}\end{bmatrix}$$

and this is the result of adding, term by term, the 3rd row of A to the second row of A.

In summary,

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix} =$$

$$= \begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21}+a_{31} & a_{22}+a_{32} & a_{23}+a_{33} & a_{24}+a_{34}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}$$

4.1.7

a. $$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{bmatrix} (a_1,a_2,a_3)\cdot(1,0,0) & (a_1,a_2,a_3)\cdot(0,1,0) & (a_1,a_2,a_3)\cdot(1,0,1) \\ (b_1,b_2,b_3)\cdot(1,0,0) & (b_1,b_2,b_3)\cdot(0,1,0) & (b_1,b_2,b_3)\cdot(1,0,1) \\ (c_1,c_2,c_3)\cdot(1,0,0) & (c_1,c_2,c_3)\cdot(0,1,0) & (c_1,c_2,c_3)\cdot(1,0,1) \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & a_2 & a_1+a_3 \\ b_1 & b_2 & b_1+b_3 \\ c_1 & c_2 & c_1+c_3 \end{bmatrix} \quad (1)$$

Equation (1) shows us that the result of multiplying A <u>on the right</u> by E_{13} was equivalent to replacing the <u>3rd</u> column of A by the term by term sum of the 3rd and the 1st <u>columns</u>.

b. Leaving the details to the interested reader (the approach closely resembles the approach of part (b) in Exercise 4.1.6), it may be shown that if A and E_{ij} are both n x n matrices, then AE_{ij} is obtained by replacing the jth column of A by the term by term sum of the ith and the jth columns, and leaving the rest of A as is.

An important observation is that E_{ij} does not commute with A. In particular, $E_{ij}A$ involves replacing the <u>ith</u> <u>row</u> of A by the term by term sum of the ith and jth <u>rows</u>; AE_{ij} involves replacing the <u>jth</u> <u>column</u> of A by the term by term sum of the ith and jth <u>columns</u>.

4.1.8

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} =$$

$$= \begin{bmatrix} (3,0,0)\cdot(a_1,b_1,c_1) & (3,0,0)\cdot(a_2,b_2,c_2) & (3,0,0)\cdot(a_3,b_3,c_3) \\ (0,3,0)\cdot(a_1,b_1,c_1) & (0,3,0)\cdot(a_2,b_2,c_2) & (0,3,0)\cdot(a_3,b_3,c_3) \\ (0,0,3)\cdot(a_1,b_1,c_1) & (0,0,3)\cdot(a_2,b_2,c_2) & (0,0,3)\cdot(a_3,b_3,c_3) \end{bmatrix}$$

4.1.8 continued

$$= \begin{bmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix} \qquad (1)$$

More generally, if

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}$$

then

$$MA = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$= \begin{bmatrix} (m,0,0)\cdot(a_1,b_1,c_1) & (m,0,0)\cdot(a_2,b_2,c_2) & (m,0,0)\cdot(a_3,b_3,c_3) \\ (0,m,0)\cdot(a_1,b_1,c_1) & (0,m,0)\cdot(a_2,b_2,c_2) & (0,m,0)\cdot(a_3,b_3,c_3) \\ (0,0,m)\cdot(a_1,b_1,c_1) & (0,0,m)\cdot(a_2,b_2,c_2) & (0,0,m)\cdot(a_3,b_3,c_3) \end{bmatrix}$$

$$= \begin{bmatrix} ma_1 & ma_2 & ma_3 \\ mb_1 & mb_2 & mb_3 \\ mc_1 & mc_2 & mc_3 \end{bmatrix} \qquad (2)$$

The result in (2) does not depend on the fact that we are dealing with 3 x 3 matrices, and accordingly, we may generalize our result rather readily to conclude that if we take the n x n diagonal matrix all of whose diagonal entries are m and multiply this by any n x n matrix A, the result is as if we simply replaced each term of A by m times the term.