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HERBERT GROSS: Hi, last time I left you dangling in suspense when I said what happens if the terms in a series are not all positive? In other words, the trouble with the last assignment was that we did quite a bit of work and yet there was a very stringent condition. Namely, that every term that you were adding happened to be a positive number. Now obviously, this need not be the case. And the question is, can you have convergence in a series in which the terms are not all positive? And what does it mean, by the way, if this is the case why we stressed the situation of positive series? And this will be the aim of today's lecture to straighten out both of these points.

At any rate, I call today's lesson 'Absolute Convergence', and I hope that the meaning of this will become clear very soon as we go along. But to answer the first question that we brought up, let's take a series in which the terms are not all positive.

Now, by the way, I will do more general things in our supplementary notes on this material. I felt though that for a blackboard illustration, I should pick a relatively straightforward example and not try to generalize it. Let's take the specific series 1 minus 1/2 plus 1/3 minus 1/4 plus 1/5, et cetera. And how do we indicate the n-th term in this case? Notice that the denominator is 'n'. The numerator oscillates between minus 1 and 1. And you see the mathematical trick to alternate signs is to raise minus 1 to a power. You see minus 1 to an even power will be positive 1. And to an odd power, negative 1. The idea here is since for 'n' equals 1 we want this to be positive, we tacked on the plus 1 here. You see in that case, 'n' being 1, 'n plus 1' is 2. Minus 1 squared is 1. And at any rate, don't be confused by this notation, it's simply a cute way all of alternating signs.

At any rate, what do we have in this particular series? We have that the terms alternate in sign. We also have that the n-th term approaches 0. Namely, the numerator alternates between plus and minus 1. The denominator is 'n'. So as 'n' increases, the terms converge on 0 as a limit. And finally, the terms keep decreasing monotonically in magnitude. In other words, forgetting about the fact that a plus outranks a minus in terms of a number line, notice that the size of 1/3 is less than the size of 1/2.

In other words, this particular series, which is called an 'alternating series', has three properties. It's called alternating because the terms alternate in sign. The n-th term approaches 0. By the way, we saw in our first lecture on series that the n-th term approaching 0 was necessary, but not sufficient for making a series converge. However, what our claim is that if in addition to this we know that the terms decrease in magnitude, our claim is that the given series will converge. In other words, what I intend to show is that 1 minus 1/2 plus 1/3, et cetera, does converge. And the way I'm going to do that is geometrically. I'm not going to try to prove this thing analytically. But as I've said before, the analytic proof virtually is just an abstraction of what we're doing over here.

Let's take a look and see what happens over here. Notice that our first term is 1. Our next term is 1/2. Our next term is 1 minus 1/2 plus 1/3, which is 5/6. What I'm driving at is that if you compute the sums, 1 minus 1/2 plus 1/3 minus 1/4, et cetera, in the order in which they're given, what you find is that the sequence of partial sums is 1, 1/2, 5/6, 7/12, 47/60, 13/20.

Now the thing that I want you to see, the reason I'm waving my hand here and why I think this helps in the lecture is look what's happening. You see because the terms alternate in sign what this means is that as I start with 's sub 1' over here, the next term will be to the left of 's sub 1'. Then the term after that will be to the right. Then to the left. Then to the right. Then to the left. Then to the right. They keep alternating this way.

Moreover, since the terms decrease in magnitude, it means that each jump is less than the jump that came immediately before. In other words, as I jumped from here to here, when I jump back I don't come back quite as far. In other words, what I'm doing now is I'm closing in. You see the odd subscripts and the even subscripts are sort of segregated. You see what's happening over here?

And finally, because the limit of the n-th term is 0, it means that this spacing-- see the difference between successive partial sums is the n-th term. That must go to 0. In other words, our limit 'L' is in here. And as 'n' increases, the squeeze is put on and we get the existence of a limit.

For example, whatever 'L' is, it must be between 13/20 and 47/60. By the way, I say this in the form of an aside. It turns out-- and for those of us who haven't seen this before, it's a very mystic result. That's why I say, who'd have guessed it? It turns out that 'L' is actually the natural log of 2. And the reason I point this out is that again, you may recall that in our notes we talked about 'Cauchy convergence', meaning what do you do when you don't know how to guess the limit? You see, the idea here is notice that this particular series converges. But we don't know what the limit is other than the fact that it's being squeezed in over here. You see here's a case where we know that a limit exists, but it's particularly difficult to explicitly name what that limit is.

But that fact notwithstanding, is it clear that because of the fact that the terms alternate, the magnitudes decrease, and the limit is 0 that these things do converge to a limit? I think it is clear. But the next question, and I apologize for what looks like slang here, but I think this is exactly what's going on in your minds right now. So what? What does this have to do with what came before and what will come later? And we're going to see again, a very, very strange thing that happens with infinite sums that does not happen with finite sums. Let me lead into that fairly gradually.

First of all, I claim that this particular series-- see, again, don't get misled by this. It's just a fancy way of saying what? 1 minus 1/2 plus 1/3, et cetera, converges. But because the plus terms cancel the minus terms, the pluses cancel the minuses, not because the terms get small fast enough. What I mean by that is this. Forget about the signs in here. Replace each term by its magnitude.

And by the way, that's where the name 'absolute' convergence is going to come from. Namely, the magnitude of a term is its absolute value. And this is what we're going to be talking about, but the idea is this. If we replace each term by its magnitude, we obtain the series. Summation 'n' goes from 1 to infinity '1 over n'. And in the last assignment, we saw in the exercises that this diverged by the integral test. In other words, if we leave out the signs, the series diverges because evidently these terms don't get small fast enough. Okay. Let me state a definition. The definition is simply this.

The series, summation 'n' goes from 1 to infinity, 'a sub n' is said to converge absolutely if the series that you get by replacing each term by its magnitude converges. Now I leave for the supplementary notes the proof that if a series converges absolutely, it converges in the first place. In other words, I think it's rather clear if you look at this thing intuitively that if I replace each term by its magnitude, I'll disregard the plus and minus signs. And that resulting series converges, then the original series must've converged also because the terms couldn't be any bigger than this. By the way, the formal proof is kind of messy in places and so this is why, as I say, I leave this for the supplementary notes. But at any rate that's what we mean by absolute convergence.

If you're given a series, you replace each term by it's magnitude. If that series converges, we call the original series absolutely convergent. Notice by the way, the tie in now between absolute convergence and positive series. Namely, by definition, the absolute value of 'a sub n' is at least as big as 0. Consequently, when we're testing for absolute convergence, the series that we test is positive. And we have tests for convergence for positive series.

Now the sequel to definition one is of course definition two, and that says what? A series which converges but not absolutely is called conditionally convergent. In other words, it converges on the condition that the sine stay exactly the way they are. An example of a conditionally convergent series is the one that we're dealing with right now. Namely, with the pluses and minuses in there, we just showed that the series converges. However, if we replace each term by its magnitude, the resulting series is summation '1 over n'. And that as we saw, diverges.

Now the question is, what's so bad about conditional convergence? What difference does it make whether a series converges absolutely or conditionally? Is there any problem that comes up? As I said before, a fantastic subtlety that occurs, a subtlety that has no parallel in our knowledge of finite arithmetic. The subtlety is this. In fact, I call it that, the subtlety of conditional convergence. Namely, the sum of a conditionally convergent series-- and this fantastic-- depends on the order in which you write the terms.

In other words, if the series converges, but conditionally if you change the order of the terms, surprising as it may seem, you actually change the sum. And the best way to do this I think at this stage, a generalization is given in the supplementary notes. But the idea is, let me just do this in terms of the problem that we were dealing with. Let's take the terms of our conditionally convergent series, 1 minus 1/2 plus 1/3, et cetera, and divide them into two teams. And if this expression bothers you, call the teams sets and that makes it much more mathematical.

Let the first set consist of the positive terms of a series. Namely, 1, 1/3, 1/5. And in general, '1 over '2n minus 1'' where 'n' is any positive whole number. The negative numbers of our team are minus 1/2-- or the set 'N', which I'll call the negative members is minus 1/2, minus 1/4, minus 1/6. In general, minus '1 over 2n'. Now again, by the integral test, we saw in our last lesson that both these series summation '1 all over '2n minus 1'' and summation '1 over 2n' diverge to infinity by the integral test.

Now what my claim is, is that because both of these two series diverge, I can now rearrange my terms to get any sum that I want.

Well, for example, suppose somebody says to me, make the sum come out to be 3/2. As I'll mention later, there's nothing sacred about 3/2. In fact, I'm mentioning that now but I'll say it again later for emphasis. I just wanted to pick a number that wouldn't be too unwieldy. But here's what the gist is. I want to make the sum at least 3/2. So what I do is I start with my positive terms from the set 'P' and add them up. 1 plus 1/3 plus 1/5, et cetera. The thing that I'm sure of is that eventually this sum must exceed 3/2. Why do I know that? Well, 'P' diverges to infinity. How could 'P' possibly diverge to infinity if the sum could never-- the sum of the terms in 'P' diverges to infinity. How could that happen if the sum never got at least as big as 3/2?

So the idea is I write down all of these terms, add them up, until my sum first exceeds or equals 3/2. And as I say, this must happen because the series diverges to infinity. Well, in particular, I observe that 1 plus 1/3 plus 1/5 is 23/15. And that's now, for the first time, bigger than 3/2.

What I do next is I annex the negative members, in other words, the members of capital 'N', until the sum falls below 3/2. Watch what I'm doing here.

I stopped at 1/5. I have 1 plus 1/3 plus 1/5. Now I subtract 1/2. That gives me 31/30 and that's less than 3/2. So now my sum is below 3/2.

What I do next is I continue with 'P' where I left off. See I left off with 1/5. I now start tacking on 1/7, 1/9, 1/11, 1/13, et cetera. Until the sum, again, exceeds 3/2. Now how do I know that the sum has to exceed 3/2? Well, remember when we add up all of the members of 'P', we get a divergent series. That means the sum increases without bound.

As we've mentioned many times so far in our course, that if you chop off a finite number of terms from a divergent series, the remaining series, what's left, still must diverge. In other words, if this series 1 plus 1/3 plus 1/5 plus 1/7 diverges to infinity, the fact that I chop off those terms that add up to just an excess of 3/2, what's left is still going to diverge to infinity. So I can keep on going this way.

What I do is I add on 1/7. The result turns out to be 247/210, which is still less than 3/2. And to spare you the gory details-- and believe me, they are gory. I worked it out myself. Without a desk calculator this gets to be a mess. It turns out that when I add on 1/13, get down to here, the sum is this, which is still less than 3/2. But then I add on 1/15. The sum gets to be this, which I simply call 'k'. That turns out to be greater than 3/2. Then you see what I do is return to my series of negative terms, tack those on till the sum falls below. And what's happening here pictorially is the following.

You see what happened was 3/2-- I added on terms till I exceeded 3/2. That was 23/15. Then I get down below 3/2. Then up again above 3/2. And still sparing you the details, notice what's happening.

Each time that I passed 3/2, I pass it by less than before because the terms are getting smaller in magnitude. What's happening is and I hope this crazy little diagram here serves the purpose. You see what's happening is I'm zeroing in on 3/2. In other words, this particular rearrangement will guarantee me that those terms will add up to 3/2. Now again, as I said before, 3/2 was not important, though the arithmetic gets messier. And again, that's the best word I can think of. In other words, I went through several sheets of paper just trying to get to the next stage over here before I realized it wasn't worth it. I mean it's something we can all do on our own. But the larger number that you choose, the more terms you're going to have to add up before you exceed this.

Don't confuse two things here. I obviously have to add up an awful lot of terms of the form 1, 1/3, 1/5, 1/7, 1/9, 1/11 to get, say a million. But the point is that since that series diverges, eventually by going out far enough-- now far enough might be billions of terms. But still a finite number, the sum will exceed 1 million. That's the key point. In other words, I can keep oscillating around any sum that I want just by exceeding it, coming back with a negative terms, getting less than that, and alternating back and forth. Again, this will be left for much greater detail for the supplementary notes and the exercises. I'll mention a little bit more about that in a few minutes. But the summary so far is this.

If the series summation 'n' goes from 1 to infinity, 'a sub n' is conditionally convergent, its limit exists. Let's not forget that. Its limit exists. But that limit depends on not changing the order in which the terms were given. In other words, the limit changes as the order of the terms is changed. That is, rearranging the terms actually changes the series. And there is nothing in finite arithmetic that is comparable to this.

In other words, if you have 50 numbers to add up, or 50 million numbers, or 50 billion numbers, no matter how you rearrange those numbers, the sum exists and is the same independently of what the rearrangement is. This is not comparable to finite arithmetic. And the moral is-- and again, I say this in slang expression because I want this to rub off. I want you to remember this. Don't monkey with conditional convergence. If the series is conditionally convergent, make sure that you add the terms in the order in which they appear. That if you change the order, you will get a different limit.

And what will happen is you'll get a limit that is the right answer to the wrong problem. In other words, changing the order of the terms changes the limit. And this is why conditional convergence is particularly annoying. It means that all of these things that come natural an ordinary arithmetic are lacking in conditional convergence.

Now, what does the sequel to this? The sequel is that all is well when you have absolute convergence. I just wrote this out to make sure that we have this in front of us.

The beauty of absolute convergence is that the sum of an absolutely convergent series is the same for every rearrangement of the terms. The details are left to the supplementary notes. Now, what am I trying to bring out by all of this?

You see, the beauty of positive series is that every time we talk about absolute convergence, the test involves a positive series. In other words, by knowing how to test positive series for convergence, we can test any series for absolute convergence.

What is the beauty of absolute convergence? The beauty of absolute convergence is that we can rearrange the terms in any order that we want if it's convenient to pick a different order than another. And the sum will not depend on this rearrangement. You see the point is we are not saying keep away from conditionally convergent series. In many important applications you have to come to grips with conditional convergence. All we are saying is that if you want to be able to fool around numerically with these series, if you don't have absolute convergence, you're in a bit of trouble.

Now you see, the point is that our textbook does a very good job in talking about absolute convergence versus conditional convergence. But for some reason, does not mention the problem of rearranging terms. And therefore, much of what I've talked about today, the importance of absolute convergence, is in terms of rearrangements. And because this material is not in the textbook, what I have elected to do is to put all of this material that we've talked about today almost verbatim except in a more generalized form, into the supplementary notes, supplying whatever proofs are necessary and whatever intuitive ideas are necessary. At any rate, read the supplementary notes, do the exercises, and we'll continue our discussion next time. And until next time, goodbye.

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