PROFESSOR: Hi, I'm Gilbert Strang, and this is the very first in a series of videos about highlights of calculus. I'm doing these just because I hope they'll be helpful. It seems to me so easy to be lost in the big calculus textbooks and the many, many problems and in the details. But do you see the big picture? Well, I hope this will help.

For me, calculus is about the relation between two functions. And one example for those two functions, one good example, is function 1, the distance, distance traveled, what you see on a trip meter in a car. And function 2, the one that goes with distance, is speed, how quickly you're going, how fast you're traveling. So that's one pair of functions.

Let me give another pair. I could get more and more, but I think if we get these two pairs, we can move forward. So in this second pair, height is function 1, how high you've climbed. If it's a graph, how far the graph goes above the axis. Up, in other words. So that's height, and then the other one tells you how fast you climb. The height tells how far you climbed. It could be a mountain. And then the slope tells you how quickly you're climbing at each point. Are you going nearly straight up? Flat? Possibly down?

So distance and speed, height and slope will serve as good examples to start with. And let me give you some letters, some algebra letters that you might use. Distance, maybe I would call that f of t. So f for how far or for function, and the idea is that t is the input. It's the time when you're asking for the distance. The output is the distance.

Or in the case of height, maybe y of x would be the right one. x is how far you go across. That's the input. And at each x, you have an output y how far up? So f is telling you how far. y is telling you the height of a graph. That's function 1, two examples of function 1.

Now, what about slope? Well, luckily, speed and slope start with the same letter, so I'll often use s for the speed or the slope for this second-- oh, it even stands for a second function. But let me tell you also the right-- the official-- letters that make the connection between function 2 and function 1. If my function is a function of time, the distance, how far I go, then the speed is-- the right letters are df dt. Everybody uses those letters. So let me say again how to pronounce: df dt. And Leibniz came up with that notation, and he just got it right.

And what would this one be? Well, corresponding to this, it looks the same, or dy dx. Again, I'll just repeat how to say that: dy dx. And that is the slope, and we have to understand what those symbols mean. Right now, I'm just writing them down as symbols.

May I begin with the most important and the simplest example of all? Let me take that case. OK, so the key example here, the one to get completely straight is the case of constant speed, constant slope. I'll just graph that. So here I'm go to graph.

Shall I make it the speed? Yeah, let's say speed. So time is going along that way. Speed is up this way. And I'm going to say in this first example that the speed is the same. We're traveling at the constant speed of let's say 40. So it stays at the height of 40. Oh, properly, I should add units like miles per hour or kilometers per hour or meters per second or whatever. For now, I'll just write 40.

OK, now if we're traveling at a speed of 40 miles per hour, what's the distance? Well, let me start with the trip meter at zero. so this is time again, and now this is going to be the distance. After one hour, my distance is 40. So if I mark t equal to 1, I've reached 40. That's height of 40. At t equal to 2, I've reached 80. At t equal to 1/2, half an hour, I've reached 20. Those points lie on a line. The graph of distance covered when you're just traveling at a steady rate, constant rate, constant speed is just a straight line.

And now I can make the connection. I've been speaking here about distance and speed. But now let me think of this as the height-- 40 is that height. 80 is that height-- and ask about slope. What is slope?

So let's just remember what's the connection here. What's the slope if that's the distance if I look at my trip meter and I know I'm traveling along at that constant speed, how do I find that speed? Well, slope, it's the distance up, which would be 40 after one hour, divided by the distance across, 40/1, or 80/2. Doesn't matter, because we're traveling at constant speed, so the slope, which is up, over, across is 40/1, 80/2, 20 over 1/2. I'll put 80/2 as one example: 40.

Oh, let me do it-- that's arithmetic. Let me do it with algebra. We don't need calculus yet, by the way. Calculus is coming pretty quickly. This is the step we can take. Because the speed is constant, we can just divide the distance by the time to find-- and this slope, let me right speed also. Up, over, across, distance over time, f/t, that gives us s. This is s.

OK, what about-- calculus goes both ways. We can go both ways here. We already have practically. Here I went in the direction from 1 to 2.

Now, I want to go in the direction-- suppose I know the speed. How do I recover the distance? If I know my speed is 40 and I know I started at zero, what's my distance? Distance or height, either one. So these are like both.

Now, I'm just going the other way. Well, you see how. How do I find f? It's s times t, right? Your algebra automatically says if you see a t there, you can put it there. So it's s times t. It's a straight line. s times t, s times x, y equal sx. Let me put another-- the same idea with my y, x letters. It's that line. In other words, if that one is constant, this one is a straight line.

OK, straightforward, but very, very fundamental. In fact, can I call your attention to something a little more? Suppose I measured between time 2 and time 1. So I'm looking between time 2 and time 1, and I look how far I went in that time. But what I'm trying is-- I'm going to put in another little symbol because it's going to be really worth knowing.

It's really the change in f divided by the change in t. I use that letter delta to indicate a difference between-- the difference between time 2 and time 1 was 1, and the difference between height 2 and height 1 was 40. You see, I'm looking at this little piece. And, of course, the slope is still 40. It's still the slope of that line. Yeah, so that really what I'm measuring in speed there, I don't always have to be starting at t equals 0, and I don't always have to be starting at f equals 0. Oh, let me draw that.

Suppose I started at f equals 40. My trip meter happened to start at 40. After an hour, I'd be up to 80. After another hour, I'd be up to 120. Do you see that this starting the trip meter, who cares where the trip meter started? It's the change in the trip meter that tells how long the trip was. Clear.

OK, so that's that example. We come back to it because it's the basic one where the speed is constant. And even if now I have to move to a changing speed, you have to let me bring calculus into these lectures.

OK, I'm going to draw another picture, and you tell me about the-- yeah, let me draw function 1, another example of function 1. So again I have time. I have distance. I'm going to start at zero, but I'm not going to keep the speed constant. I'm going to start out at a good speed, but I'm going to slow down. Do you see me slowing down there? I don't mean slowing down with the chalk. I mean slowing down with slope. The slope started out steep. By here, by that point, the slope was zero.

What was the car doing here? The car is certainly moving forward because the distance is increasing. Here it's increasing faster. Here it's increasing barely. In other words, we're putting on the brakes. The car is slowing down. We're coming to a red light. In fact, there is the red light right at that time.

Now, just stay with it to think what would the speed look like for this problem? If that's a picture of the function, just let's get some idea. I'm not going to have a formula yet. I'm not putting in all the details. Well, actually, I don't plan to put in all the details of calculus of every possible step we might take. It's the important ones I'm hoping to show you and I'm hoping for you to see that they are important.

OK, what is important? Roughly, what does the graph looks like? Well, the speed-- the slope-- started out somewhere up there. Yeah, it started out at a good speed and slowed down. And by this point, ha! Let's mark that time here on that graph. Do you see what is the speed at that moment? The speed at that moment is zero. The car has stopped. The speed is decreasing. Let me make it decrease, decrease, decrease, decrease, and at that moment, the speed is zero right there. That's that point.

See, two different pictures, two different functions. but same information. So calculus has the job of given one of those functions, find the other one. Given this function, find that one. This way is called-- from function one to function two, that's called differential calculus. Big, impressive word anyway. That's function one to two, finding the speed.

Going the other direction is called integral calculus. The step is called integration when you take the speed over that period of time, and you recover the distance. So it's differential calculus in one direction, integral calculus in the other. Now, here's a question. Let me continue that curve a little longer. I got it to the red light. Now imagine that the distance starts going down from that point. What's happening?

The distance is decreasing. The car is going backwards. It's going in reverse. The speed, what's the speed? Negative. The speed, because distance is going from higher to lower, that counts for negative speed. The speed curve would be going down here. Do you see that that's a not brilliantly drawn picture, but you're seeing the-- that's the farthest it went. Then the car started backwards, and the speed curve reflected that by going below zero. You see, two different curves, but same information.

I'm remembering an old movie. I don't know if you saw an old B movie called *Ferris Bueller's Day Off.* Did you see that? So the kid had borrowed his father's-- not borrowed, but lifted his father's good car and drove it a lot like so and put on a lot of mileage. The trip meter was way up, and he knew his father was going to notice this. So he had the idea to put the car up on a lift, put it in reverse, and go for a while, and the trip meter would go backwards.

I don't know if trip meters do go backwards. It's kind of tough to watch them while going in reverse. But if whoever made the car understood calculus, as you do, the speedometer-- now that I think of it, speedometers don't have a below zero. They should have. And trip meters should go backwards.

I mean, that movie was just made for a calculus person. Maybe I'm remembering more. I think it didn't work or something. And the kid got mad and kicked the car, and it fell off the lift, went through the glass window. Anyway, calculus would have saved him if only the car had been-- or the meters in the car had been made correctly.

All right, that's one pair. That's our first real pair in which the speed changes. OK. I thought in this first video, later, even today, I'll get to a case where we have formulas. That's what calculus moves into. When f of t is given by some formula, well, here it's given by a formula: s times t. A simple formula. And then, knowing that, we know that the speed is s. Later, we got more functions. But let me take an example, just because these pairs of functions are everywhere.

What could I take? Maybe height of a person. Height of a person. OK, so this is now another example, just to get practice in the relation between the height of a person and the rate of change of the height. So this is the height. Maybe I'll call it y. Let me write height of a person. And what is this going to be? What is function two? Well, slope doesn't seem quite right. The point about function two is it tells how fast function one changes. It's the rate of change of the height. It's the rate of change. So let me call it s, and it'll be the rate of change. Good if I use those words.

Yeah, so I want to think just how we grow, a typical person growing. In fact, as I wrote this on the board, I thought of another pair. Can I just say it in words, this other pair, and then I'll come back to this one?

Here's another pair. This could be money in a bank. Wealth sounds better. Let's call it wealth. That's zippier. And then what is this one? If this is your wealth, your total assets, what's your worth? This would be the rate of change, how quickly you're saving. s could be for saving. Or if you're down here, s is for spending, right? If s is positive, that means you're wealth is increasing, you're saving. Negative s means you're spending, and your wealth goes whatever, maybe-- I hope-- up.

Height is mostly up, right? So let me come back to height of a person. Now, where-- oh, and this is time in years. This is t in years, and this, too, of course. Actually, I realize you started at t equals zero: birth. You do start at a certain-- actually, what do I know? You don't say tall. You say long. But then as soon as you can stand up, it's tall, so let's say tall.

Shall we guessed 20 inches? If that's way off, I apologize to everybody. Let me just say 20, 20 inches. OK, at year zero. OK, and then presumably you grow. OK, so you grow a little. What are we headed for? About 60, 70 inches or something. Anyway, you grow. Let's say that's 10 years old and here is 20 years old. OK, so you grow. Maybe you grow faster than that. Let's say you're a healthy person here. OK, up you grow.

And then at about maybe age 12 or 13, there's a growth spurt. And maybe the point is, how do we see that growth spurt on the two graphs? Differently, but it's the same growth spurt. OK, so here your height suddenly jumps up. Boy, yeah, you catch up with everybody. And then at about 12 or 13 well, then unfortunately, it doesn't do that forever, and it kind of levels off here. It levels off, and actually you don't grow a whole lot more. In fact, I think when you get to about-- oh, I don't know. Whatever. We won't discuss this point. I say when you get too old, you probably lose some. Let's not emphasize that.

OK, so here is the-- now, what's happening over here? Well, it's the slope of that graph. So the slope might be-- this is time zero, but you're growing right away. The s graph, the rate of growth graph, doesn't start at zero. It starts how fast you're growing, whatever you're growing, whatever that slope is. It's fantastic that when we draw graphs of things, the word "slope" is suddenly the right word.

OK, so you're growing, maybe at a pretty good rate here. And let me mark out 10 and 20 years. And OK, you're doing well, you're coming along here, and then the growth spurt. OK, so then suddenly, your rate of growth takes off. But it doesn't stay that way, right? Your rate of growth levels off, in fact, levels way off, levels-- you'll come down to here, and you probably don't grow a lot. Do you see the two? This was the growth curve. This was the fast growth. But then it stopped. Up here, it slowed down. Here, it dropped. And oh, if we allow for this person who lived too long, height actually drops.

OK, there is an example in which I don't-- also I'm sure people have devised approximate formulas for average growth rates, but you see, I'm not-- it's the idea of the relation between function one and function two that I'm emphasizing.

Now, my last example, let me take one more example, one more example for this first lecture. So let me take a case in which the speed is-- so here will be my two-- let's use speed. Let's use this as distance. This is distance again, and graph two, as always, will be speed. And I'm going to take a case in which it's given by a formula. I'm going to let the speed be increasing steadily.

OK, so my speed graph this time is going to go up at a constant rate. So this is the speed s. This is the time t. So this would be s equals-- s is proportional to t. That's where you get a straight line. s is let's say a times t. That a, a physicist, if we were physicists, would say acceleration. You're accelerating. You're keeping your foot on the gas, steadily speeding up, and so then s is proportional to t.

Now, think about the distance. What's happening with distance? If this is accelerating, you're going faster and faster. You're covering more and more speed, more and more distance, more and more quickly. If this is slope, the slope is increasing. Look, the graph-- let's start the trip meter at zero.

So you started with a speed of zero. You were not really increasing distance until you got slightly beyond zero, and then it slightly started to increase. But then it increases faster and faster, right? It never gets infinitely fast, but it keeps going upwards. And the calculus question would be can we give a formula-- an equation-- for the distance? Because in this case, I guess I started with function two, and therefore, it's function one that I want to look at. It's always pairs of functions.

OK, now, let's think where this would actually happen. If we were leaning over the Tower of Pisa or whatever, like Galileo, and drop something, or even just drop something anywhere, that would be-- we drop it. At the beginning, it has no speed, but of course, instantly it picks up speed. The a would have something to do with the gravitational constant for the Earth, whatever, and then maybe-- yeah. And what would be the distance?

OK, now can I just mention a small miracle of calculus? A small miracle. I'm going this direction now from speed to distance so I'm doing integral calculus. And we'll get to that later. In the first lectures, we're almost always going from one to two. But here is a neat fact about going from two to one, that if this is the time t, then, of course, this height here will be a times t.

And the amazing fact is that this graph tells you the area under this one. Graph one-- function one-- tells you the area under the graph two. And in this example with a nice constant acceleration, steady increase in speed, we know this. This is a triangle. It has a base of t. It has a height of at. And the area of a triangle, of course, the area, and my point is that the area is function one-- amazing; that's just terrific-- will be-- the area of this triangle here is 1/2 of the base times the height. That's the area, and calculus will tell us that's function one. So this function is 1/2 of a times t squared.

So there is a function one, and here is df dt. If I go back to the first letters that I mentioned, if this is my function f, then this is my function that-- and notice what kind of a curve that is. Do you recognize that with a square? That tells me it's a parabola, a famous and important curve. And, of course, it's important because it has such a neat formula.

OK, so we have found the function one. We've recovered the information in that lost black box, the distance box, the trip meter, from what we did find in black box two, the speed, the record of speed. And notice, I'm using the speed all the way from here to here. The speed kind of tells me how the distance is piled up. The distance is kind of a running total, where the speed at that moment is an instant thing. Oh, we have to do that in future lectures. The difference between a running total of total distance covered and a speed that's telling me at a moment, at an instant how distance is changing. The slope at this very point t, that slope is this height, is at.

OK, so there you have the first-- well, I'll say the second. The first pair of calculus was this one. f equals st, and it's derivative was s. Our second pair is f is this, and will you allow me to write df dt? If f is 1/2 of at squared, then df dt is at. You'll see this rule again. The power two dropped to a power one. But the two multiplied the thing so it canceled the 1/2 and just left the a.

OK, that's a start on the highlights of calculus. Thanks.

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