

PROFESSOR: OK, hi. This is the second in my videos about the main ideas, the big picture of calculus. And this is an important one, because I want to introduce and compute some derivatives. And you'll remember the overall situation is we have pairs of functions, distance and speed, function 1 and function 2, height of a graph, slope of the graph, height of a mountain, slope of a mountain.

And it's the connection between those two that calculus is about. And so our problem today is, you could imagine we have an airplane climbing. Its height is y as it covers a distance x . And its flight recorder will-- Well, probably it has two flight recorders. Let's suppose it has. Or your car has two recorders.

One records the distance, the height, the total amount achieved up to that moment, up to that time, t , or that point, x . The second recorder would tell you at every instant what the speed is. So it would tell you the speed at all times. Do you see the difference? The speed is like what's happening at an instant.

The distance or the height, y , is the total accumulation of how far you've gone, how high you've gone. And now I'm going to suppose that this speed, this second function, the recorder is lost. But the information is there, and how to recover it. So that's the question. How, if I have a total record, say of height-- I'll say mostly with y of x . I write these two so that you realize that letters are not what calculus is about. It's ideas.

And here is a central idea. if I know the height-- as I go along, I know the height, it could go down-- how can I recover from that height what the slope is at each point? So here's something rather important, that's the notation that Leibniz created, and it was a good, good idea for the derivative. And you'll see where it comes from.

But somehow I'm dividing distance up by distance across, and that ratio of up to across is a slope. So let me develop what we're doing. So the one thing we can do and now will do is, for the great functions of calculus, a few very special, very important functions. We will actually figure out what the slope is.

These are given by formulas, and I'll find a formula for the slope, dy/dx equals. And I won't write it in yet. Let me keep a little suspense. But this short list of the great functions is tremendously valuable. The process that we go through takes a little time, but once we do it it's done. Once we write in the answer here, we know it.

And the point is that other functions of science, of engineering, of economics, of life, come from these functions by multiplying-- I could multiply that times that-- and then I need a product rule, the rule for the derivative, the slope of a product. I could divide that by that. So I need a quotient rule.

I could do a chain. And you'll see that's maybe the best and most valuable. e to the sine x , so I'm putting e to the x together with sine x in a chain of functions, e to the sine of x . Then we need a chain rule. That's all coming.

Let me start-- Well, let me even start by giving away the main facts for these three examples, because they're three you want to remember, and might as well start now. The x to the n th, so that's something if n is positive. x to the n th climbs up. Let me draw a graph here of y equals x squared, because that's one we'll work out in detail, y equals x squared.

So this direction is x . This direction is y . And I want to know the slope. And the point is that that slope is changing as I go up. So the slope depends on x . The slope will be different here. So it's getting steeper and steeper. I'll figure out that slope.

For this example, x squared went in as 2, and it's climbing. If n was minus 2, just because that's also on our list, n could be negative, the function would be dropping. You remember x to the minus 2, that negative exponent means divide by x squared. x to the minus 2 is 1 divided by x squared, and it'll be dropping. So n could be positive or negative here. So I tell you the derivative.

The derivative is easy to remember, the set number n . It's another power of x , and that power is one less, one down. You lose one power. I'm going to go through the steps here for n is equal to 2, so I hope to get the answer 2 times x to the 2 minus 1 will just be 1, $2x$. But what does the slope mean? That's what this lecture is really telling you.

I'll tell you the answer for if it's sine x going in, beautifully, the derivative of sine x is cos x , the cosine. The derivative of the sine curve is the cosine curve. You couldn't hope for more than that. And then we'll also, at the same time, find the derivative of the cosine curve, which is minus the sine curve. It turns out a minus sine comes in because the cosine curve drops at the start.

And would you like to know this one? e to the x , which I will introduce in a lecture coming very soon, because it's the function of calculus. And the reason it's so terrific is, the connection between the function, whatever it is, whatever this number e is to whatever the x th power means, the slope is the same as the function, e to the x . That's amazing. As the function changes, the slope changes and they stay equal.

Really, my help is just to say, if you know those three, you're really off to a good start, plus the rules that you need. All right, now I'll tackle this particular one and say, what does slope mean? So I'm given the recorder that I have. This is function 1 here. This is function 1, the one I know. And I know it at every point.

If I only had the trip meter after an hour or two hours or three hours, well, calculus can't do the impossible. It can't know, if I only knew the distance reached after each hour, I couldn't tell what it did over that hour, how often you had to break, how often you accelerated. I could only find an average speed over that hour. That would be easy.

So averages don't need calculus.

It's instant stuff, what happens at a moment, what is that speedometer reading at the moment x , say, x equal 1. What is the slope? Yeah, let me put in x equals 1 on this graph and x equals 2. And now x squared is going to be at height 1. If x is 1, then x squared is also 1. If x is 2, x squared will be 4. What's the average? Let me just come back one second to that average business. The average slope there would be, in a distance across of 1, I went up by how much? 3. I went up from 1 to 4. I have to do a subtraction. Differences, derivatives, that's the connection here.

So it's 4 minus 1. That is 3. So I would say the average is $3/1$. But that's not calculus. Calculus is looking at the instant thing. And let me begin at this instant, at that point.

What does the slope look like to you at that point? At x equals 0, here's x equals 0, and here's y equals 0. We're very much 0. You see it's climbing, but at that moment, it's like it just started from a red light, whatever. The speed is 0 at that point. And I want to say the slope is 0. That's flat right there. That's flat.

If I continued the curve, if I continued the x squared curve for x negative, it would be the same as for x positive. Well, it doesn't look very the same. Let me improve that. It would start up the same way and be completely symmetric. Everybody sees that, at that 0 position, the curve has hit bottom.

Actually, this will be a major, major application of calculus. You identify the bottom of a curve by the fact that the slope is 0. It's not going up. It's not going down. It's 0 at that point. But now, what do I mean by slope at a point?

Here comes the new idea. If I go way over to 1, that's too much. I just want to stay near this point. I'll go over a little bit, and I call that little bit Δx . So that letter, delta, signals to our minds small, some small. And actually, probably smaller than I drew it there. And then, so what's the average? I'd like to just find the average speed, or average slope.

If I go over by Δx , and then how high do I go up? Well, the curve is y equals x squared. So how high is this? So the average is up first, divided by across. Across is our usual Δx . How far did it go up?

Well, if our curve is x squared and I'm at the point, Δx , then it's Δx squared. That's the average over the first piece, over short, over the first piece of the curve. Out is-- from here out to Δx . OK.

Now, again, that's still only an average, because Δx might have been short. I want to push it to 0. That's where calculus comes in, taking the limit of shorter and shorter and shorter pieces in order to zoom in on that instant, that moment, that spot where we're looking at the slope, and where we're expecting the answer is 0, in this case. And you see that the average, it happens to be especially simple. Δx squared over Δx is just

delta x.

So the average slope is extremely small. And I'll just complete that thought. So the instant slope-- instant slope at 0, at x equals 0, I let this delta x get smaller and smaller. I get the answer is 0, which is just what I expected it. And you could say, well, not too exciting. But it was an easy one to do. It was the first time that we actually went through the steps of computing.

This is a, like, a delta y. This is the delta x. Instead of $3/1$, starting here I had delta x squared over delta x. That was easy to see. It was delta x. And if I move in closer, that average slope is smaller and smaller, and the slope at that instant is 0. No problem.

The travel, the climbing began from rest, but it picked up speed. The slope here is certainly not 0. We'll find that slope. We need now to find the slope at every point. OK. That's a good start.

Now I'm ready to find the slope at any point. Instead of just x equals 0, which we've now done, I better draw a new graph of the same picture, climbing up. Now I'm putting in a little climb at some point x here. I'm up at a height, x squared. I'm at that point on the climb. I'd like to know the slope there, at that point. How am I going to do it?

I will follow this as the central dogma of calculus, of differential calculus, function 1 to function 2. Take a little delta x, go as far as x plus delta x. That will take you to a higher point on the curve. That's now the point x plus delta x squared, because our curve is still y equals x squared in this nice, simple parabola. OK.

So now I you look at distance across and distance up. So delta y is the change up. Delta x is the across. And I have to put what is delta y. I have to write in, what is delta y? It's this distance up. It's x plus delta x squared. That's this height minus this height. I'm not counting this bit, of course. It's that that I want. That's the delta y, is this piece.

So it's up to this, subtract x squared. That's delta y. That's important. Now I divide by delta x. This is all algebra now. Calculus is going to come in a moment, but not yet.

For algebra, what do I do? I multiply this out. I see that thing squared. I remember that x is squared. And then I have this times this twice. $2x$ delta x's, and then I have delta x squared. And then I'm subtracting x squared. So that's delta y written out in full glory.

I wrote it out because now I can simplify by canceling the x squared. I'm not surprised. Now, in this case, I can actually do the division. Delta x is just there. So it leaves me with a $2x$. Delta x over delta x is 1. And then here's a delta x squared over a delta x, so that leaves me with one delta x. As you get the hang of calculus, you see the important things is like this first order, delta x to the first power. Delta x squared, that, when divided by delta x,

gives us this, which is going to disappear. That's the point.

This was the average over a short but still not instant range, distance. Now, what happens? Now dy/dx . So if this is short, short over short, this is darn short over darn short. That d is, well, it's too short to see. So I don't actually now try to separate a distance dy . This isn't a true division, because it's effectively $0/0$. And you might say, well, $0/0$, what's the meaning? Well, the meaning of $0/0$, in this situation, is, I take the limit of this one, which does have a meaning, because those are true numbers. They're little numbers but they're numbers.

And this was this, so now here's the big step, leaving algebra behind, going to calculus in order to get what's happening at a point. I let Δx go to 0. And what is that? So Δy over Δx is this. What is the dy/dx ? So in the limit, ah, it's not hard. Here's the $2x$. It's there. Here's the Δx . In the limit, it disappears.

So the conclusion is that the derivative is $2x$. So that's function two. That's function two here. That's the slope function. That's the speed function. Maybe I should draw it. Can I draw it above and then I'll put the board back up?

So here's a picture of function 2, the derivative, or the slope, which I was calling s . So that's the s function, against x . x is still the thing that's varying, or it could be t , or it could be whatever letter we've got. And the answer was $2x$ for this function.

So if I graph it, it starts at 0, and it climbs steadily with slope 2. So that's a graph of s of x . And for example-- yeah, so take a couple of points on that graph-- at x equals 0, the slope is 0. And we did that first. And we actually got it right. The slope is 0 at the start, at the bottom of the curve.

At some other point on the curve, what's the slope here? Ha, yeah, tell me the slope there. At that point on the curve, an average slope was $3/1$, but that was the slope of this, like, you know-- sometimes called a chord. That's over a big jump of 1. Then I did it over a small jump of Δx , and then I let Δx go to 0, so it was an instant infinitesimal jump.

So the actual slope, the way to visualize it is that it's more like that. That's the line that's really giving the slope at that point. That's my best picture. It's not Rembrandt, but it's got it. And what is the slope at that point? Well, that's what our calculation was.

It found the slope at that point. And at the particular point, x equals 1, the height was 2. The slope is 2. The actual tangent line is only-- is there. You see? It's up. Oh, wait a minute. Yeah, well, the slope is 2. I don't know. This goes up to 3. It's not Rembrandt, but the math is OK.

So what have we done? We've taken the first small step and literally I could say small step, almost a play on

words because that's the point, the step is so small-- to getting these great functions. Before I close this lecture, can I draw this pair, function 1 and function 2, and just see that the movement of the curves is what we would expect.

So let me, just for one more good example, great example, actually, is let me draw. Here goes x . In fact, maybe I already drew in the first letter, lecture that bit out to 90 degrees. Only if we want a nice formula, we better call that π over 2 radians. And here's a graph of sine x . This is y . This is the function 1, sine x .

And what's function 2? What can we see about function 2? Again, x . We see a slope. This is not the same as x squared. This starts with a definite slope. And it turns out this will be one of the most important limits we'll find.

We'll discover that the first little Δx , which goes up by sine of Δx , has a slope that gets closer and closer to 1. Good. Luckily, cosine does start at 1, so we're OK so far. Now the slope is dropping.

And what's the slope at the top of the sine curve? It's a maximum. But we identify that by the fact that the slope is 0, because we know the thing is going to go down here and go somewhere else. The slope there is 0. The tangent line is horizontal. And that is that point. It passes through 0. The slope is dropping. So this is the slope curve.

And the great thing is that it's the cosine of x . And what I'm doing now is not proving this fact. I'm not doing my Δx 's. That's the job I have to do once, and it won't be today, but I only have to do it once. But today, I'm just saying it makes sense the slope is dropping. In that first part, I'm going up, so the slope is positive but the slope is dropping.

And then, at this point, it hits 0. And that's this point. And then the slope turns negative. I'm falling. So the slope goes negative, and actually it follows the cosine. So I go along here to that point, and then I can continue on to this point where it bottoms out again and then starts up.

So where is that on this curve? Well, I'd better draw a little further out. This bottom here would be the-- This is our $\pi/2$. This is our π , 180 degrees, everybody would say. So what's happening on that curve? The function is dropping, and actually it's dropping its fastest. It's dropping its fastest at that point, which is the slope is minus 1.

And then the slope is still negative, but it's not so negative, and it comes back up to 0 at $3\pi/2$. So this is the point, $3\pi/2$. And this has come back to 0 at that point. And then it finishes the whole thing at 2π . This finishes up here back at 1 again. It's climbing.

All right, climbing, dropping, faster, slower, maximum, minimum, those are the words that make derivatives important and useful to learn. And we've done, in detail, the first of our great list of functions. Thanks.

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