# STUDY GUIDE TO CALCULUS

This Student Study Guide accompanies the textbook *Calculus* by Gilbert Strang. It is correlated section by section with the essential points of the text. The Guide contains four components which experience has shown are most helpful:

1. Model problems with comments and complete solutions.

2. Extra drill problems included with exercises for chapter review.

3. Read-through questions from the text with the blanks filled in.

4. Solutions to selected even-numbered problems in each section.

The Guide can be obtained from bookstores or by writing directly to Wellesley–Cambridge Press. Our address is Box 812060, Wellesley MA 02181 (prepayment by check). The press also publishes two other basic textbooks by Gilbert Strang: *Introduction to Linear Algebra* and *Introduction to Applied Mathematics*. This Study Guide shares the same goal, to teach mathematics in an active and purposeful way.

1.1 Velocity and Distance

The first step in calculus (in my opinion) is to begin working with *functions*. The functions can be described by formulas or by graphs. The first graphs to study are straight lines, and the functions that go with them are linear: $y = mx + b$ or (using other letters) $f = vt + C$. It takes practice to connect the formula $y - 4 = 7(x - 2)$ to the straight line with slope 7 going through the point (2,4). You need to be able to rewrite that formula as $y = 7x - 10$. You also need to be able to find the formula from the graph. Here is an example: *Find the equations for these straight lines.*
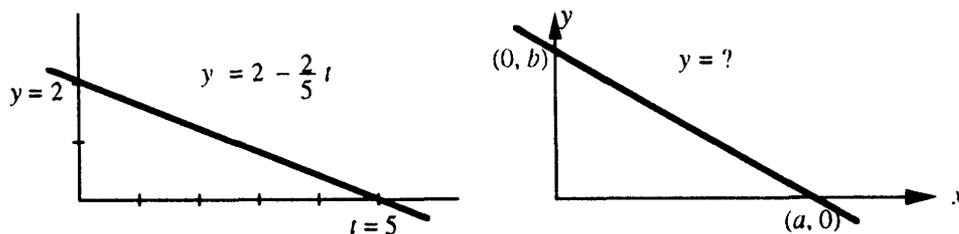
The first line goes down 2 and across 5. The slope is $\frac{-2}{5} = -.4$. The minus sign is because y decreases as t increases – the slope of the graph is negative. The "starting point" is $y = 2$ when $t = 0$. The equation is $y = 2 - .4t$. Check that this gives $y = 0$ when $t = 5$, so the formula correctly predicts that second point.

Problem 1: *Find the equation for the second line.* Comment: The points $(0, b)$ and $(a, 0)$ are given by letters not numbers. This is typical of mathematics, which is not so much about numbers as most people think – it is really about patterns. Sometimes numbers help, other times they get in the way. Many exercises in mathematics books are really asking you to find the pattern by solving a general problem (with letters) instead of a special problem (with numbers). But don't forget: You can always substitute 5 and 2 for the letters a and b.

Solution to Problem 1: The slope is $-\frac{b}{a}$. The equation is $y = b - \frac{b}{a}x$. Compare with $y = 2 - \frac{2}{5}t$.

Pairs of functions. Calculus deals with two functions at once. We have to understand both functions, and *how they are related.* In writing the book I asked myself: Where do we find two functions? What example can we start with? The example should be familiar and it should be real – not just made up. The best examples build on what we already know. The first pair of functions is distance $f(t)$ and velocity $v(t)$.

Comment: This example is taken from *life*, not from physics. Don't be put off by symbols like $f(t)$ and $v(t)$. The velocity is certainly connected to the distance traveled. This connection is clearest when the velocity is constant. If you go 120 miles in 2 hours at steady speed, then $v = \frac{120 \text{ miles}}{2 \text{ hours}} = 60$ miles per hour. That example converts to straight line graphs: The distance graph goes up with slope 60. Starting from $f(0) = 0$, the equation for the line is $f(t) = 60t$.

Most people see how $f = 60t$ leads to $v = 60$. Now look at the connection between $f(t)$ and $v(t)$ in the next graphs.
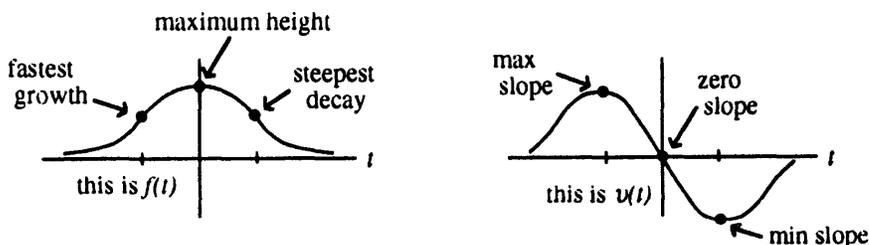

What do you see? The slope of f is zero at the center. Thus $v = 0$ where f is a maximum. The maximum of v is where the graph of f is steepest (upwards). Then v is largest (positive). When the graph of f flattens out, v drops back toward zero.

With formulas I could specify these functions exactly. The distance might be $f(t) = \frac{1}{1+t^2}$. Then Chapter 2 will find $\frac{-2t}{(1+t^2)^2}$ for the velocity $v(t)$. Very often calculus is swept up by formulas, and the ideas get lost. You need to know the rules for computing $v(t)$, and exams ask for them, but it is not right for calculus to turn into pure manipulations. Our goal from the start is to *see the ideas.*

Again comes the question: Where to go after $f = 60t$ and $v = 60$? The next step is to allow two velocities (two slopes). The velocity jumps. The distance graph switches from one straight line to another straight line – with a different slope. It takes practice to write down the formulas and draw the graphs. Here are six questions about straight line graphs that change slopes. The figure shows $f(t)$, the questions are about $v(t)$.

Answers are given for the first graph. Test yourself on the second graph.

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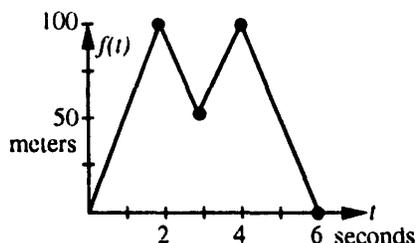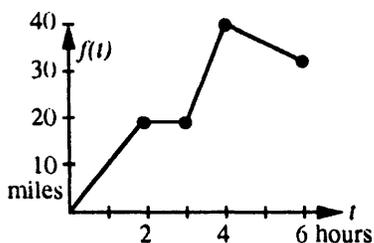
2   During the first two hours, the car travels _____ miles. The velocity during this time is _____ miles per hour. The slope of the distance graph from $t = 0$ to $t = 2$ is _____.
Answer: Miles traveled = 20, velocity = 10 miles per hour, slope = $\frac{20}{2} = 10$.

3   From $t = 2$ to $t = 3$ the car travels _____ miles. The velocity during this time is _____ miles per hour. The slope of this segment is _____. Answer: 0,0,0.

4   From $t = 3$ to $t = 4$ the car travels _____ miles. The velocity during this time is _____ mph. The slope of the f-graph from $t = 3$ to $t = 4$ is _____. Answer: $20, 20, \frac{40-20}{4-3} = 20$.

5   From $t = 4$ to $t = 6$, the car travels *backwards* _____ miles. The change in f is _____ miles. The velocity is _____ miles per hour. The slope of this segment of the f-graph is _____. Answer: Backwards 10 miles, change in f is -10, velocity -5 miles per hour, slope $\frac{30-40}{6-4} = -5$.

6   *Draw graphs of $v(t)$ from those two graphs of $f(t)$.*
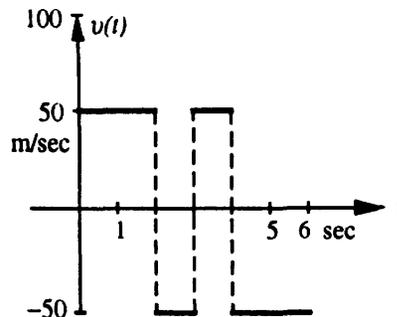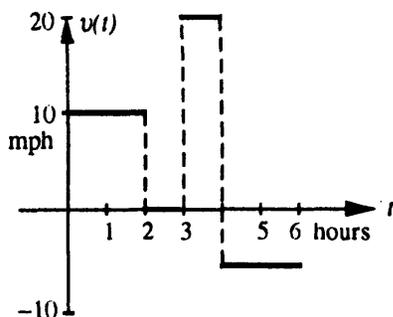
**7**   Give a 4-part formula for $f(t)$. Each piece is a line segment described by a linear function $f(t) = vt + C$. The first piece is just $f(t) = vt$ because it starts from $f(0) = 0$.
   - Use the form $vt + C$ if you know the slope v and the height C when $t = 0$. With different letters this form is $y = mx + b$. Use $f(t) = f(3) + v(t - 3)$ if you know v and you also know the distance $f(3)$ at the particular time $t = 3$. Section 2.3 on *tangent lines* uses both forms. The second form becomes $y = f(a) + f'(a)(x - a)$ and the slope is denoted by $f'(a)$.

From $t = 3$ to $t = 4$, our slope is 20 and the starting point is $(3,20)$. The formula for this segment is $f(t) = 20 + 20(t - 3), 3 \le t \le 4$. This simplifies to $f(t) = 20t - 40$. The complete solution is:

$$f(t) = \begin{cases} 10t + 0, & 0 \le t \le 2 \\ 20, & 2 \le t \le 3 \\ 20 + 20(t - 3), & 3 \le t \le 4 & \text{or} & 20t - 40 \\ 40 - 5(t - 4), & 4 \le t \le 6 & \text{or} & -5t + 60 \end{cases}$$

IMPORTANT If we are given $v(t)$ we can discover $f(t)$. But we do need to know the distance at one particular time like $t = 0$. The table of f's below was started from $f(0) = 0$. Questions **8–11** allow you to fill in the missing parts of the table. Then the piecewise constant $v(t)$ produces a piecewise linear $f(t)$.

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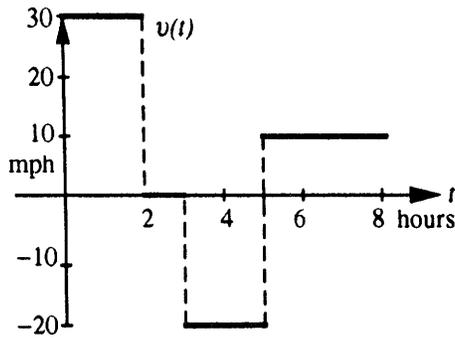
t	$f(t)$
0	0
2	60
3	60 + 0
5	60 + 0 + _____
8	60 + 0 + (−40)
	+ _____ = _____
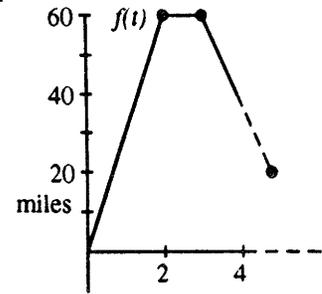
8 From $t = 0$ to $t = 2$, the velocity is _____ mph. The distance traveled during this time is _____ miles. The area of this rectangular portion of the graph is _____ × _____ = _____. Answer: 30, 60, 30 × 2 = 60.

9 From $t = 2$ to $t = 3$, the velocity is _____ miles per hour. The "area" of this portion of the graph (a rectangle with no height) is _____ × _____ = _____. Answer: $0, 1 \times 0 = 0$.

10 From $t = 3$ to $t = 6$, the velocity is _____ mph. During this time the vehicle travels *backward* _____ miles. The area of this rectangular portion is _____ × _____ = _____. Keep in mind that area below the horizontal axis counts as *negative*. Answer: $-20, 40, -20 \times 2 = -40$.

11 From $t = 5$ to $t = 8$ the velocity is _____ mph. During this time the vehicle travels forward _____ miles. The area of this rectangular portion is _____ × _____ = _____.
Answer: 10, 30, 10 ×3 = 30.

12 Using the information in **8-11**, finish the table and complete the distance graph out to time $t = 8$. Connect the points $(t, f(t))$ in the table with straight line segments.

13 What is the distance traveled from $t = 0$ to $t = 4$? From $t = 0$ to $t = 7$?
Answer: The points (4, 40) and (7, 40) can be read directly from the graph. The distance in each case is 40 miles.
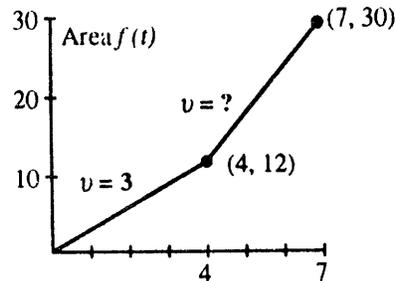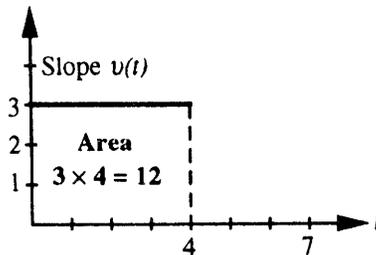
14 This is Problem 1.1.32 in the text. Suppose $v = 3$ up to time $t = 4$. What constant velocity will lead to $f(7) = 30$ if $f(0) = 0$? Give graphs and formulas for $v(t)$ and $f(t)$.

 • We know that $f = 12$ when $t = 4$ because distance = rate × time = $3 \times 4 = 12$. Note that the slope of the f-graph from $t = 0$ to $t = 4$ is 3, the velocity. Since we want $f(7) = 30$, we must go $30 - 12 = 18$ more miles. The slope of the second piece of the f-graph should be $\frac{30-12}{7-4} = \frac{18}{3} = 6$. The desired velocity is $v = 6$. The equation of this segment is $f(t) = 12 + 6(t - 4)$. At the breakpoint $t = 4$, the velocity $v(4)$ is *not* defined, but the distance $f(4)$ *is* defined. The symbol in $v(t)$ is < while the symbol in $f(t)$ is ≤.

4

$$v(t) = \begin{cases} 3, & 0 < t < 4 \\ 6, & 4 < t < 7 \end{cases} \qquad f(t) = \begin{cases} 3t + 0, & 0 \le t \le 4 \\ 12 + 6(t-4), & 4 \le t \le 7. \end{cases}$$

Here are solutions to the read-through questions and selected even-numbered exercises for Section 1.1.

Starting from $f(0) = 0$ at constant velocity v, the distance function is $f(t) =$ **vt**. When $f(t) = 55t$ the velocity is $v =$ **55**. When $f(t) = 55t + 1000$ the velocity is still **55** and the starting value is $f(0) =$ **1000**. In each case v is the slope of the graph of f. When **v(t)** is negative, the graph of **f(t)** goes downward. In that case area in the v-graph counts as **negative**.

Forward motion from $f(0) = 0$ to $f(2) = 10$ has $v =$ **5**. Then backward motion to $f(4) = 0$ has $v = $ **−5**. The distance function is $f(t) = 5t$ for $0 \le t \le 2$ and then $f(t)$ equals **5(4 − t)** (not −5t). The slopes are **5** and **−5**. The distance $f(3) =$ **5**. The area under the v-graph up to time 1.5 is **7.5**. The domain of f is the time interval $0 \le \mathbf{t} \le \mathbf{4}$, and the range is the distance interval $0 \le \mathbf{f} \le \mathbf{10}$. The range of $v(t)$ is only **5 and −5**.

The value of $f(t) = 3t + 1$ at $t = 2$ is $f(2) =$ **7**. The value 19 equals $f(\mathbf{6})$. The difference $f(4) - f(1) =$ **9**. That is the change in distance, when $4 - 1$ is the change in **time**. The ratio of those changes equals **3**, which is the slope of the graph. The formula for $f(t) + 2$ is $3t + 3$ whereas $f(t + 2)$ equals **3t+7**. Those functions have the same slope as f: the graph of $f(t) + 2$ is shifted up and $f(t + 2)$ is shifted to the **left**. The formula for $f(5t)$ is **15t+1**. The formula for $5f(t)$ is **15t + 5**. The slope has jumped from 3 to **15**.

The set of inputs to a function is its **domain**. The set of outputs is its **range**. The functions $f(t) = 7 + 3(t - 2)$ and $f(t) = vt + C$ are **linear**. Their graphs are straight lines with slopes equal to **3** and **v**. They are the same function, if $v =$ **3** and $C =$ **1**.

10 $v(t)$ is negative-zero-positive; $v(t)$ is above 55 then equal to 55; $v(t)$ increases in jumps; $v(t)$ is zero then positive. All with corresponding $f(t)$.

26 The function increases by 2 in one time unit so the slope (velocity) is 2; $f(t) =$ **2t + C** with constant $C = f(0)$.

36 At $t = 0$ the reading was $.061 + 10(.015) =$ **.211**. A drop of $.061 - .04 = .021$ would take $.021/.015$ hours. This was the Exxon Valdez accident in Alaska.

1.2   Calculus Without Limits   (page 14)

This section was rewritten for the second printing of the book, in order to bring out the central ideas. One idea starts with numbers like $f = 3, 5, 9$. Their differences are 2 and 4. The sum of those differences is $2 + 4 = 6$. This equals $9 - 3$, or $f_{\text{last}} - f_{\text{first}}$. (It is like a card trick, where you think of numbers and only tell me the differences. Then I tell you the last number minus the first number.) The middle number 5 cancels out:

$$(5 - 3) + (9 - 5) = 9 - 3 \quad \text{or} \quad (f_1 - f_0) + (f_2 - f_1) = f_2 - f_0.$$

Taking sums is the "inverse" of taking differences. This is an early pointer to calculus.

• The *"derivative"* of $f(t)$ is the slope $v(t)$. This is like differences.

• The *"integral"* of $v(t)$ is the area $f(t)$. This is like sums.

Teaching question: *Why mention these ideas so early?* They are not proved. They can't even be explained in complete detail. Here is my answer: You the student should know where the course is going. I believe in seeing an idea several times, instead of "catch it once or lose it forever". Here was a case where a few numbers give a preview of the Fundamental Theorem of Calculus.

As the idea unfolds and develops, so does the notation. At first you have specific numbers 3,5,9. Then you have general numbers f_0, f_1, f_2. Then you have a piecewise linear function. Then you have a general function $f(t)$.

Similarly: At first you have differences $5 - 3$ and $9 - 5$. Then you have $v_1 = f_1 - f_0$ and $v_j = f_j - f_{j-1}$. Then you have a piecewise constant function. Then you have the derivative $v(t) = \frac{df}{dt}$.

The *equation for a straight line* is another idea that unfolds and develops. It was already in some of the problems for Section 1.1. Now Section 1.2 brings it out again, especially the income tax example. The tax in the second income bracket is

$$f(x) = \text{ tax on } \$20,350 + (\text{ tax rate } .28)(\text{income over } \$20,350).$$

This is so typical of what comes later: $f(x) = f(a) + (\text{slope})(x - a)$. Do those letters x and a make the equation less clear? For many students I believe they do, at least at first. The number 20,350 is definite where the letter a is vague. The number .28 is specific where the slope $f'(a)$ will be general. A teacher is always caught by this dilemma. Mathematics expresses the pattern by symbols, but it is understood first for numbers.

The best way is to see the same idea several times, which we do for straight lines. Section 2.3 finds the tangent line to a curve. Section 3.1 uses the slope for a linear approximation. Section 3.7 explains Newton's method for solving $f(x) = 0$ – replace the curve by its tangent line. Will you see that the underlying idea is the same? I hope so, because it is the key idea of differential calculus: *Lines stay close to curves* (at least for a while).

1   Suppose $v = 3$ for $t < T$ and $v = 1$ for $t > T$. If $f(0) = 0$, find formulas for $f(t)$.

  - The equation of the first segment is $f(t) = 0 + 3t$. This continues to $t = T$ when $f = 3T$. Using the fact that $(T, 3T)$ starts the second segment and $v = 1$ is the slope, we find $f(t) = 3T + 1(t - T)$ for $T \le t$.

2   Suppose f_0, f_1, f_2, f_3 are the distances 0, 8, 12, 14 at times $t = 0, 1, 2, 3$. Find velocities v_1, v_2, v_3 and a formula that fits v_j. Graph $v(t)$ and $f(t)$ with constant and linear pieces.

  - The differences are $v_1 = 8, v_2 = 4$, and $v_3 = 2$. The velocities are halved at each step: $v_1 = \frac{1}{2}(16), v_2 = \frac{1}{4}(16), v_3 = \frac{1}{8}(16)$. The formula $v_j = (\frac{1}{2})^j(16)$ or $v_j = 16/2^j$ fits all velocities.

3   Using the differences $v_1, v_2, \cdots, v_j$ and the starting value f_0, give a formula for f_j.

  - $f_j = f_0 + v_1 + v_2 + \cdots + v_j$. *You should notice this formula.*

4   Suppose the tax rates are increased by 2 percentage points to .17, .30, and .33. At the same time, a tax credit of \$500 is allowed. If your original income was x, does your tax $f(x)$ go up or down after the changes? What if the \$500 is only a deduction from income?

  - The answer depends on x. At $x = \$10,000$ the rate increase costs only 2% of \$10,000, which is \$200. You win. At $x = \$25,000$ the 2% rate increase balances the \$500 credit.

    If the \$500 is *deducted from income* of \$10,000, then you pay 17% of \$9,500 instead of 15% of \$10,000. This time you lose. To find the income at which deduction balances the 2% rate increase, solve $.15x = .17(x - 500)$ to find $x = \$4,250$.

5 (Algebra) If the distances are $f_j = j^2$, show that the velocities are $v_j = 2j - 1$.

- Since the difference v_j is $f_j - f_{j-1}$, you have to subtract $(j-1)^2$ from j^2 :

$$j^2 - (j-1)^2 = j^2 - (j^2 - 2j + 1) = 2j - 1.$$

Please notice the summary at the end of Section 1.2, and then go over these read-through problems. Here are the blanks filled in, followed by even-numbered solutions.

Start with the numbers $f = 1, 6, 2, 5$. Their differences are $v = $ **5, −4, 3**. The sum of those differences is **4**. This is equal to f_{last} minus f_{first}. The numbers 6 and 2 have no effect on this answer because in $(6 - 1) + (2 - 6) + (5 - 2)$ the numbers 6 and 2 **cancel**. The slope of the line between $f(0) = 1$ and $f(1) = 6$ is **5**. The equation of that line is $f(t) = \mathbf{1 + 5t}$.

With distances 1, 5, 25 at unit times, the velocities are **4 and 20**. These are the **slopes** of the f-graph. The slope of the tax graph is the tax **rate**. If $f(t)$ is the postage cost for t ounces or t grams, the slope is the **cost per ounce (or per gram)**. For distances 0, 1, 4, 9 the velocities are **1, 3, 5**. The sum of the first j odd numbers is $f_j = \mathbf{j^2}$. Then f_{10} is **100** and the velocity v_{10} is **19**.

The piecewise linear sine has slopes **1, 0, −1, −1, 0, 1**. Those form a piecewise **constant** cosine. Both functions have **period** equal to 6, which means that $f(t + 6) = \mathbf{f(t)}$ for every t. The velocities $v = 1, 2, 4, 8, \ldots$ have $v_j = \mathbf{2^{j-1}}$. In that case $f_0 = 1$ and $f_j = \mathbf{2^j}$. The sum of 1, 2, 4, 8, 16 is **31**. The difference $2^j - 2^{j-1}$ equals $\mathbf{2^{j-1}}$. After a burst of speed V to time T, the distance is **VT**. If $f(T) = 1$ and V increases, the burst lasts only to $T = 1/V$. When V approaches infinity, $f(t)$ approaches a **step** function. The velocities approach a **delta** function, which is concentrated at $t = 0$ but has area **1** under its graph. The slope of a step function is **zero or infinity**.

8 $f(t) = \mathbf{1 + 10t}$ for $0 \le t \le \frac{1}{10}$, $f(t) = \mathbf{2}$ for $t \ge \frac{1}{10}$

10 $f(3) = \mathbf{12}; g(f(3)) = g(12) = \mathbf{25}; g(f(t)) = g(4t) = \mathbf{8t + 1}$. When t is changed to $4t$, distance increases **four** times as fast and the velocity is multiplied by 4.

28 The second difference $f_{j+1} - 2f_j + f_{j-1}$ equals $(f_{j+1} - f_j) - (f_j - f_{j-1}) = v_{j+1} - v_j = 4$.

32 The period of $v + w$ is **30**, the smallest multiple of both 6 and 10. Then v completes five cycles and w completes three. An example for functions is $v = \sin \frac{\pi x}{3}$ and $w = \sin \frac{\pi x}{5}$.

42 The ratios $\frac{\Delta f}{\Delta t}$ are $\frac{e-1}{1} = 1.718$ and $\frac{e^{.1}-1}{.1} = 1.052$ and $\frac{e^{.01}-1}{.01} = 1.005$. They are approaching 1.

1.3 The Velocity at an Instant (page 21)

Worked examples 1-4 bring out the key ideas of Section 1.3. You are computing the slope of a curve – genuine calculus! To compute this slope at a particular time $t = 1$, find the change in distance $f(1 + h) - f(1)$. To compute the velocity at a general time t, find the change $f(t + h) - f(t)$. In both cases, divide by h for average slope. Let $h \to 0$ for instantaneous velocity = slope at point.

1 For $f(t) = t^2 - t$ find the *average* speed between (a) $t = 1$ and $t = 2$ (b) $t = 0.9$ and $t = 1.0$ (c) $t = 1$ and $t = 1 + h$. (d) Use part (c) as $h \to 0$ to find the *instantaneous* speed at $t = 1$. (e) What is the *average* speed from t to $t + h$? (f) What is the formula for $v(t)$?

- (a) $f(2) = 2^2 - 2 = 2$ and $f(1) = 1^2 - 1 = 0$. Average speed $= \frac{2-0}{2-1} = 2$
- (b) $f(0.9) = 0.81 - 0.9 = -0.09$. Average speed is $\frac{f(1) - f(0.9)}{1 - (0.9)} = \frac{.09}{0.1} = 0.9$

- (c) $\frac{f(1+h)-f(1)}{(1+h)-(1)} = \frac{[(1+h)^2-(1+h)]-[1^2-1]}{(1+h)-1} = \frac{[(1+2h+h^2)-(1+h)]-0}{h} = \frac{h^2+h}{h} = h+1.$

- (d) Let h get extremely small. Then $h+1$ is near 1. The speed at $t=1$ is $v=1$.

- (e) $\frac{f(t+h)-f(t)}{(t+h)-t} = \frac{[(t+h)^2-(t+h)]-[t^2-t]}{h} = \frac{t^2+2ht+h^2-t-h-t^2+t}{h}.$
  After cancelling this is

$$\frac{h^2+2ht-h}{h} = h+2t-1 = average\ speed.$$

- (f) Let h go to zero. The average $h+2t-1$ goes to $0+2t-1$. Thus $v(t)=2t-1$. *Check* $v=1$ at $t=1$.

2   If $v(t)=6-2t$, find a formula for $f(t)$ and graph it.

- $f(t)$ is the area under the v-graph from 0 to t. For small t, this is the area of a *trapezoid* with heights 6 and $6-2t$. The base is t. Since area = average height times base, $f(t) = \frac{1}{2}(6+6-2t)t = (6-t)t = 6t-t^2$. At $t=3$ we reach $f(3)=18-9=9$. This is the area of the triangle.

  Does this area formula $f(t)=6t-t^2$ continue to apply after $t=3$, when some area is negative? *Yes it does.* The triangle beyond $t=3$ has base $t-3$ and height (or depth) $6-2t$. Its area is $\frac{1}{2}bh = \frac{1}{2}(t-3)(6-2t) = -9+6t-t^2$. After adding the positive area 9 of the first triangle, we still have $6t-t^2$.

  The negative height automatically gave negative area. The graph is a parabola that turns down at $t=3$, when new area is negative.
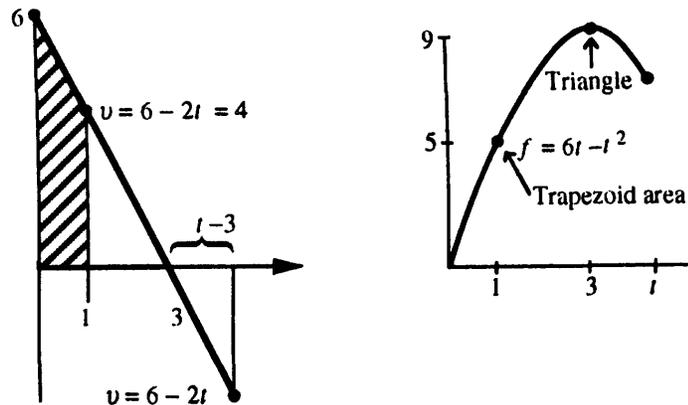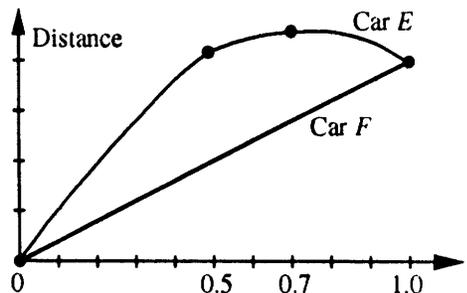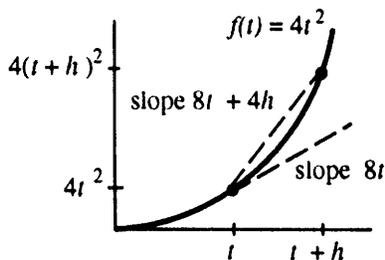

3   If $f(t)=4t^2$, find the slope at time t. Start with the average from t to $t+h$.

- Step 1: The average slope is

$$\frac{f(t+h)-f(t)}{(t+h)-t} = \frac{4(t+h)^2-4t^2}{h} = \frac{(4t^2+8th+4h^2)-4t^2}{h}$$

This simplifies to $\frac{8th+4h^2}{h} = \mathbf{8t+4h}.$

- Step 2: Decide what happens to the average slope as h approaches zero. The average $8t + 4h$ approaches the slope at a point (the instantaneous velocity) which is $8t$.

4 The graph shows distances for two cars. E and F start and end at the same place (why?) but their trips are different. Car F travels at a steady velocity. Describe what is happening to car E at these times:

(a) $t = 0$ to $t = .5$ (b) $t = .5$ (c) $t = .5$ to $t = .7$ (d) $t = .7$ (e) $t = .7$ to $t = 1$.

- (a) Car E covers more ground than F. It is going faster than F.

- (b) Since the slope of the graph *at this instant* is the same for E and F, their speed is the same.

- (c) E gains less ground than F, so E is going slower than F.

- (d) *At this instant E has speed zero.* The distance graph is level. *Maximum distance, zero speed.*

- (e) Car E loses ground, the slope of the distance graph is negative: E is going backward.

Here are the read-throughs and selected solutions to even-numbered problems.

Between the distances $f(2) = 100$ and $f(6) = 200$, the average velocity is $\frac{100}{4} = \mathbf{25}$. If $f(t) = \frac{1}{4}t^2$ then $f(6) = \mathbf{9}$ and $f(8) = \mathbf{16}$. The average velocity in between is $\mathbf{3.5}$. The instantaneous velocities at $t = 6$ and $t = 8$ are $\mathbf{3}$ and $\mathbf{4}$.

The average velocity is computed from $f(t)$ and $f(t+h)$ by $v_{\text{ave}} = \frac{1}{h}(\mathbf{f(t + h) - f(t)})$. If $f(t) = t^2$ then $v_{\text{ave}} = \mathbf{2t + h}$. From $t = 1$ to $t = 1.1$ the average is $\mathbf{2.1}$. The instantaneous velocity is the **limit** of v_{ave}. If the distance is $f(t) = \frac{1}{2}at^2$ then the velocity is $v(t) = \mathbf{at}$ and the acceleration is $\mathbf{a}$.

On the graph of $f(t)$, the average velocity between A and B is the slope of **the secant line**. The velocity at A is found by **letting B approach A**. The velocity at B is found by **letting A approach B**. When the velocity is positive, the distance is **increasing**. When the velocity is increasing, the car is **accelerating**.

2 (c) $\frac{\frac{1}{2}a(t^2 + 2th + h^2) - \frac{1}{2}at^2}{h} = at + \frac{1}{2}ah$. The limit at $h = 0$ is $v = at : = $ (acceleration) $\times$ (time).

(e) $\frac{6 - 6}{h} = 0$ (limit is 0); (f) the limit is $v(t) = 2t$ (and $f(t) = t^2$ gives $\frac{(t+h)^2 - t^2}{h} = 2t + h$).

14 True (the slope is $\frac{\Delta f}{\Delta t}$); False (the curve is partly steeper and partly flatter than the secant line which gives the average slope); True (because $\Delta f = \Delta F$); False (V could be larger in between).

18 The graph is a parabola $f(t) = \frac{1}{2}t^2$ out to $f = 2$ at $t = 2$. After that the slope of f stays constant at 2.

20 Area to $t = 1$ is $\frac{1}{2}$; to $t = 2$ is $\frac{3}{2}$; to $t = 3$ is 2; to $t = 4$ is $\frac{3}{2}$; to $t = 5$ is $\frac{1}{2}$; area from $t = 0$ to $t = 6$ is zero. The graph of $f(t)$ through these points is parabola-line-parabola (symmetric)-line-parabola to zero.

26 $f(t) = t - t^2$ has $v(t) = 1 - 2t$ and $f(3t) = 3t - 9t^2$. The slope of $f(3t)$ is $3 - 18t$. This is **3v(3t)**.

28 To find $f(t)$ multiply the time t by the average velocity. This is because $v_{\text{ave}} = \frac{f(t) - f(0)}{t - 0} = \frac{f(t)}{t}$.

1.4   Circular Motion   (page 28)

This section is important but optional (if that combination is possible). It is important because it introduces sines and cosines, and it also introduces the idea of a "parameter" t. No other example could do this better. These are *periodic* functions, which repeat every time you go around the circle. For any class studying physics, circular motion is essential. For calculus the section is optional, because we return in Section 2.4 to compute the derivatives of $\sin t$ and $\cos t$ from $\Delta f / \Delta t$.

It is certainly possible to study *circular motion* and omit up-and-down *harmonic motion*. In the circular case the ball is at $x = \cos t, y = \sin t$. In the up-and-down case the shadow is still at height $y = \sin t$, but now $x = 0$. My idea was that this up-and-down motion is made easy by its connection to circular motion. When we know the velocity of the ball on the circle, we know the derivatives of $\sin t$ and $\cos t$.

The standard motion has speed 1. The velocity vector is tangent to the unit circle of radius 1. The derivative of $\sin t$ is $\cos t$, and the derivative of $\cos t$ is $-\sin t$. Every student is going to learn those rules – by seeing the circle, you see what they mean. Calculus concentrates so much on a few particular functions (amazingly few), and sines and cosines are on that short list. I am not in favor of "late trigonometry," because seeing a function several times is the way to understand it.

Conclusion: This section *introduces* $\sin t$ and $\cos t$. *Don't study them to death.* Just begin to see how they work.

Questions **1-4** refer to a ball going counterclockwise around the unit circle, centered at $(0,0)$ with radius 1. The speed is 1 (say 1 meter/sec to have units). Assume the ball starts at angle zero. Then $x = \cos t$ and $y = \sin t$.

1   How long does it take for 10 revolutions? This means 10 times around the circle.

- The distance around a circle is $2\pi r$, which is 2π meters since $r = 1$. One revolution at speed 1 m/sec takes 2π seconds. So 10 revolutions take 20π seconds.

2   At time $t = \pi$, where is the ball?

- The ball is halfway around the circle at $x = -1, y = 0$.

3   Where is the ball (approximately) at $t = 14$ seconds?

- $\frac{14}{2\pi} \approx 2.23$, so the ball has made about $2\frac{1}{4}$ revolutions. Its position after that $\frac{1}{4}$ revolution is near the top of the circle where $x = 0, y = 1$. More accurately $y = \sin 14 = .99$.

4   Suppose that string is cut at $t = \frac{5\pi}{3}$. When and where would the ball hit the x axis?

- You need to know some trigonometry, which is shown in the figure. The ball is at $B = \left(\cos \frac{5\pi}{3}, \sin \frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. When the string is cut, the ball travels along BP at one meter per second. The tangent of the angle $POB = \frac{\pi}{3}$ is $\frac{PB}{OB} = \frac{\sqrt{3}}{1}$. Therefore the ball goes from B to P in $\sqrt{3}$ seconds. The time of arrival is $\frac{5\pi}{3} + \sqrt{3}$.

  The right triangle has $OB^2 + BP^2 = 1^2 + (\sqrt{3})^2 = 4$, so the hypotenuse is $OP = 2$.
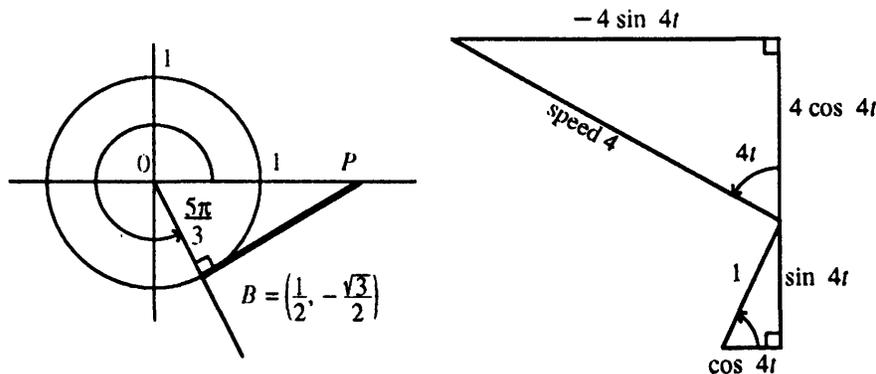
5   Another ball travels the unit circle counterclockwise with speed 4 m/sec starting at angle 0.

(a) What angle does it reach at time t? Answer: On the unit circle, a distance of 1 on the circumference corresponds to a central angle of 1 radian. Since the speed is 4 m/sec the central angle is increasing at 4 radians/sec. *At time t the angle is $4t$ radians.*

(b) What are the ball's x and y coordinates at time t? Answer: $(x, y) = (\cos 4t, \sin 4t)$.

(c) What are the ball's x and y velocities at time t? Answer: The velocity (the tangent vector) is drawn with length 4 because the speed is 4. The triangle inside the circle is similar to the triangle outside the circle. The sides of the larger triangle are four times longer. Thus the vertical velocity is $4\cos 4t$ and the horizontal velocity is $-4\sin 4t$.

  - Summary: Motion around the unit circle with speed k (counterclockwise) gives $x = \cos kt$ and $y = \sin kt$. The velocity is $-k\sin kt$ (horizontal) and $k\cos kt$ (vertical).

6   What are the components of velocity if a ball travels at 1 radian per second around a circle of radius R feet (starting at angle 0)?

- The ball's position at time t is $(R\cos t, R\sin t)$. Since the ball travels R feet along the circumference for each radian, the speed is R ft/sec. The horizontal velocity is $-R\sin t$ and the vertical velocity is $R\cos t$.

Motion around a circle of radius R feet at k radians per second gives $x = R\cos kt$ and $y = R\sin kt$. The speed is kR ft/sec. The horizontal velocity is $-Rk\sin kt$. The vertical velocity is $Rk\cos kt$.

11

Here are read-throughs and selected solutions.

A ball at angle t on the unit circle has coordinates $x = \cos t$ and $y = \sin t$. It completes a full circle at $t = 2\pi$. Its speed is **1**. Its velocity points in the direction of the **tangent**, which is **perpendicular** to the radius coming out from the center. The upward velocity is **cos t** and the horizontal velocity is **− sin t**.

A mass going up and down level with the ball has height $f(t) = \sin t$. This is called simple **harmonic** motion. The velocity is $v(t) = \cos t$. When $t = \pi/2$ the height is $f = 1$ and the velocity is $v = 0$. If a speeded-up mass reaches $f = \sin 2t$ at time t, its velocity is $v = 2 \cos 2t$. A shadow traveling *under* the ball has $f = \cos t$ and $v = -\sin t$. When f is distance = area = integral, v is **velocity = slope = derivative**.

14 The ball goes halfway around the circle in time π. For the mass to fall a distance 2 in time π we need $2 = \frac{1}{2}a\pi^2$ so $a = 4/\pi^2$.

16 The area is $f(t) = \sin t$, and $\sin \frac{\pi}{6} - \sin 0 = \frac{1}{2}$.

18 The area is still $f(t) = \sin t$, and $\sin \frac{3\pi}{2} - \sin \frac{\pi}{2} = -1 - 1 = -2$.

20 The radius is 2 and time is speeded up by 3 so the speed is 6. There is a minus sign because the cosine starts downward (ball moving to left).

26 Counterclockwise with radius 3 starting at $(3,0)$ with speed 12.

1.5   Review of Trigonometry   (page 33)

Right triangles show the usual picture of trigonometry. The sine and cosine and tangent are ratios of the sides. The cosecant and secant and cotangent are the same ratios turned upside down.

A circle shows a better picture of trigonometry. The angle goes all the way to 360° and beyond. Of course we change 360° to 2π radians. Then distance around the circle is $r\theta$ = radius times angle. The derivative of $\sin t$ is $\cos t$, when angles are in radians. Otherwise we would have factors $\frac{2\pi}{360}$ which nobody wants. This section is helpful as a quick reference to the laws of trigonometry. *Don't forget:*

$$\sin 2t = 2 \sin t \cos t \quad \text{and} \quad \cos 2t = 2\cos^2 t - 1 = 1 - 2\sin^2 t.$$

Three more to remember: $\sin^2\theta + \cos^2\theta = 1$ and $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \csc^2\theta$. Those all come from $x^2 + y^2 = r^2$, when you divide by r^2 then x^2 then y^2.

The distance from $(2,5)$ to $(3,3)$ is $\sqrt{1^2 + 2^2} = \sqrt{5}$. The book proves the addition formulas $\cos(s - t) = \cos s \cos t + \sin s \sin t$ and after sign change $\cos(s + t) = \cos s \cos t - \sin s \sin t$.

When $s = t$ you get $2t$, the "double-angle": $\cos 2t = \cos^2 t - \sin^2 t$ or $2\cos^2 t - 1$. Therefore $\frac{1}{2}(1 + \cos 2t) = \cos^2 t$, a formula needed in calculus.

1 (Problem 1.5.8) Find the distance from $(1,0)$ to $(0,1)$ along (a) a straight line (b) a quarter-circle and (c) a semicircle centered at $(\frac{1}{2}, \frac{1}{2})$.

- (a) Straight distance $\sqrt{1^2 + 1^2} = \sqrt{2}$. (b) The central angle is $\frac{\pi}{2}$, so the distance is $r\theta = \frac{\pi}{2}$.
- (c) The central angle is π and the radius is $\frac{\sqrt{2}}{2}$, half the length in part (a). Distance $\frac{\sqrt{2}}{2}\pi$.
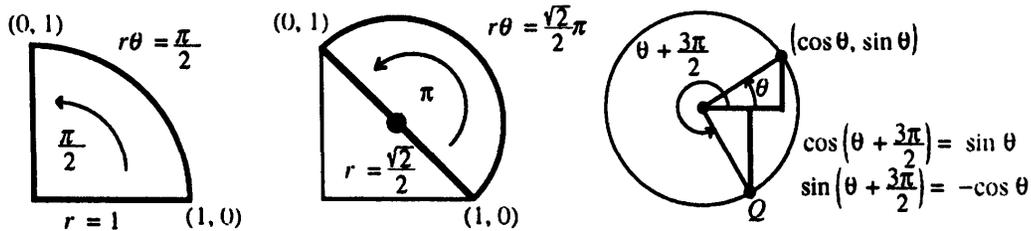
2 Simplify $\sin(\theta + \frac{3\pi}{2})$. Use formula (a), then use the circle.

- $\sin\theta\cos\frac{3\pi}{2} + \cos\theta\sin\frac{3\pi}{2}$ equals $(\sin\theta)(0) + (\cos\theta)(-1) = -\cos\theta$.
- Graphical solution: $\theta + \frac{3\pi}{2}$ is $\frac{3}{4}$ of a rotation ahead of θ. From the picture you see that $\sin(\theta + \frac{3\pi}{2})$ is the y coordinate of Q. This is $-\cos\theta$.

3 Find every angle θ that satisfies $\cos\theta = -1$.

- We know that $\cos\pi = -1$. Since the cosine repeats when θ is increased or decreased by 2π, the answer is $\theta = \pi \pm n(2\pi)$ for $n = 0, 1, 2, \cdots$. This gives the set $\theta = \pm\pi, \pm 3\pi, \pm 5\pi, \cdots$.

Read-throughs and selected even-numbered solutions :

Starting with a **right** triangle, the six basic functions are the **ratios** of the sides. Two ratios (the cosine x/r and the **sine** y/r) are below 1. Two ratios (the secant r/x and the **cosecant** r/y) are above 1. Two ratios (the **tangent** and the **cotangent**) can take any value. The six functions are defined for all angles θ, by changing from a triangle to a **circle**.

The angle θ is measured in **radians**. A full circle is $\theta = 2\pi$, when the distance around is $2\pi r$. The distance to angle θ is θr. All six functions have period 2π. Going clockwise changes the sign of θ and $\sin\theta$ and $\tan\theta$. Since $\cos(-\theta) = \cos\theta$, the cosine is **unchanged (or even)**.

4 $\cos 2(\theta + \pi)$ is the same as $\cos(2\theta + 2\pi)$ which is $\cos 2\theta$. Since $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$, this also has period π.

14 $\sin 3t = \sin(2t+t) = \sin 2t \cos t + \cos 2t \sin t$. This equals $(2\sin t \cos t)\cos t + (\cos^2 t - \sin^2 t)\sin t$ or $3\sin t - 4\sin^3 t$.

26 $\sin\theta = \theta$ at $\theta = 0$ **and never again**. Reason: The right side θ has slope 1 and the left side has slope $\cos\theta < 1$. (So the graphs of $\sin\theta$ and θ can't meet a second time.)

30 $A\sin(x+\phi)$ equals $A\sin x\cos\phi + A\cos x\sin\phi$. That expression must match with $a\sin x + b\cos x$. Thus $a = A\cos\phi$ and $b = A\sin\phi$. Then $a^2 + b^2 = A^2\cos^2\phi + A^2\sin^2\phi = A^2$. Therefore $\mathbf{A} = \sqrt{\mathbf{a^2 + b^2}}$ and $\tan\phi = \frac{A\sin\phi}{A\cos\phi} = \frac{b}{a}$.

34 The amplitude and period of $2\sin\pi x$ are both **2**.

1.6   A Thousand Points of Light   (page 34)

These two pages are to explain the figures on the back cover. They are not for studying. If **Mathematica** is available, or any other way to draw these point-graphs, there are many that you could experiment with. Try $y = \sin cn$ for different choices of c and different ranges of n. The graphs are surprising. The **Mathematica** command is ListPlot [Table[Sin [cn], {n,N}]]. Put in c and N.

1.7   Computing in Calculus   (page 36)

Every year brings progress in three key directions. All three are essential, as computer labs and computer assignments become an accepted part of calculus courses. Without any claim to completeness, here are notes on recent activity.

1   *Graphing Calculators*   The TI-85 and HP-48 have extended the range of the TI-81 and HP-28. For subjects up to and including calculus, the TI-81 (least powerful of the four) is still a good choice. The more powerful HP's can be computing tools also in advanced courses. Clemson University is a leader in teaching with HP's, and calculators will soon be allowed on College Board exams. You may be familiar with them before starting calculus.

I find that quick calculations in class bring out the meaning of symbols. An example is already in the exercises for Section 1.2 – the computation of $(2^{.1} - 1)/.1$ and $(2^{.01} - 1)/.01$. The expression $[f(x + \Delta x) - f(x)]/\Delta x$ and also the function $f(x) = 2^x$ can be encountered early and actively.

2   *Calculus Laboratories*   These are becoming widespread. The goal is to add a valuable new experience, but to avoid overburdening students or faculty. The start-up effort is considerable. In the long run, it will pay off.

> DERIVE is one popular choice of software. At Duke University it is combined with MathCAD. Other universities have selected MAPLE (for example RPI), and they find it increasingly accessible and convenient. A third strong choice is **Mathematica** (at Georgia Tech). At this writing **Mathematica** is not offered in an inexpensive student version. MATLAB has just taken that step – this is the premier code for matrix calculations (with graphics).
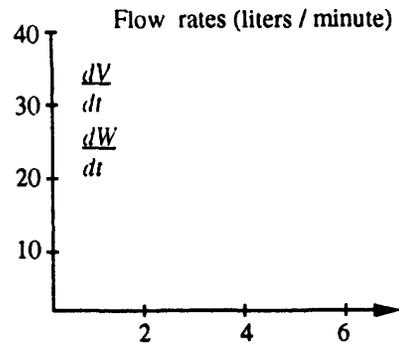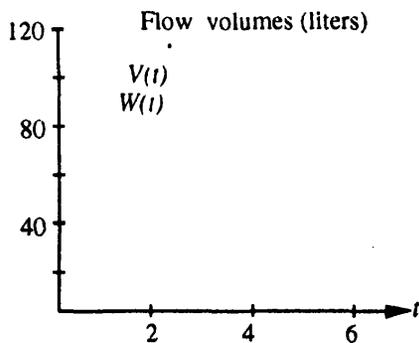
> The rate of progress with software needs to extend to laboratory manuals. It is still difficult to identify outstanding sources of problems – which should go beyond a translation of old calculus questions into computer programs. At its best, a calculus lab can be a welcome addition, provided *subtraction* is also allowed – all of us are less bound by the syllabus than we think.

3   *Projects and Activities*   Projects take 1-2 weeks, activities take 1–2 days. An essential part of each project is a report. Work in groups is demonstrated as very successful – the problem of equal work for equal grade is not severe in actual practice, and the advantages are great.

Ithaca College and New Mexico State have been pioneers with projects. My favorite in-class activity is this *Water Tank Problem*, which can come early to emphasize slopes and graphs. It was created by Peter Taylor, adapted at Ithaca College, and is slightly extended here. Teachers and students are invited to work through it together, on the board and *in the class*.

At time $t = 0$, water begins to flow into an empty tank at the rate of 40 liters/minute. This flow rate is held constant for two minutes. From $t = 2$ to $t = 4$, the flow rate is gradually reduced to 5 liters/minute. This rate is constant for the final two minutes. At $t = 6$ the tank contains 120 liters.

A Draw a graph of the volume $V(t)$ of water in the tank for $0 \le t \le 6$.

B What is the average rate of flow into the tank over the entire six-minute period? Show how this can be interpreted on your graph.

C Starting at time $t = 2$, water is pumped out at the constant rate of 15 liters/minute. Now $V(t)$ above represents the total amount of water that has flowed *in*, and we let $W(t)$ represent the total amount pumped *out*. Plot $W(t)$ on the same axes. Show how to interpret the volume of water in the tank at any time.

D (*This is calculus*) Find on your graph the point at which the water level in the tank is a *maximum*. What is the instantaneous flow rate from the hose into the tank at this time?

E Graph the *rates* of flow in and out, for $0 \le t \le 6$. Mark the point where the water level is a maximum. Extra question: Can the flow rate into the tank be linear from $t = 2$ to $t = 4$?

Flow volumes (liters)

120 · $V(t)$ $W(t)$ 80 40 2 4 6 t

Flow rates (liters / minute)

40 $\dfrac{dV}{dt}$ 30 $\dfrac{dW}{dt}$ 20 10 2 4 6 t

1   Chapter Review Problems

Graph Problems   These questions are asking for sketches by hand, not for works of art. It is surprising how much of calculus shows up in these pictures.

G1 Draw $f(x) = \sin x$ between $x = 0$ and $x = 4\pi$. Mark all maximum points by M and all minimum points by m. Mark the points where the slope is steepest (going up or down) by S.
Under that graph draw $v(x) = \cos x$ between $x = 0$ and $x = 4\pi$. *Line it up correctly*. Repeat the letters M, m, S down on the cosine graph. Notice that M and m are at points where the cosine is _____, and S is at points where the cosine is _____.

G2 Draw $y = 2 - x^2$ between $x = -1$ and $x = 1$. Mark the maximum point by M and the steepest points by S. Directly under that graph draw $v = -2x$. Move the letters M and S down to this slope graph. When y has a maximum its slope is _____. When y is steepest its slope has a _____.

G3 Draw a smooth curve that goes up then down then up again. A specific choice would be the graph of $y = x^3 - x$ from $x = -2$ to $x = 2$. Mark the maximum, minimum, and steepest points by M, m, S. Under your curve draw a graph of its slope. (The slope of $x^3 - x$ is $3x^2 - 1$.) If the graphs are lined up, then M and m are at points where the slope is zero.
New question: What is different about the M point and the m point? How can you tell a maximum from a minimum, looking at the "zero crossing" on the slope graph?
Answer: At a maximum of $y(x)$, its slope crosses from _____ to _____.

G4   **The triangles of trigonometry.** Draw a right triangle with sides marked $\cos\theta$, $\sin\theta$, and 1. Enlarge it to a triangle with sides marked 1, $\tan\theta$, and $\sec\theta$. (You multiplied the sides of the first triangle by $\sec\theta$, which is greater than 1.) From the second triangle read off the identity $1 + \tan^2\theta = $ _____. Now draw a third triangle with sides $\cot\theta$, 1, and $\csc\theta$. You have multiplied the first triangle by $\csc\theta$. This is $1/\sin\theta$ and again it is greater than 1. From this third triangle read off the identity $1 + \cot^2\theta = $ _____.
Question: Is the third triangle always larger than the second triangle? *Answer: No.* To get the third triangle from the second, multiply by $\frac{\cos\theta}{\sin\theta}$. This is larger than 1 when θ is larger than _____. The triangles have $0 < \theta < 90°$. At $\theta = 45°$, which two triangles are the same?

Computing Problems   Use a computer or calculator. You do not need a supercomputer.

C1   Find the tax on incomes of $x_1 = \$1,000,000$ and $x_2 = \$2,000,000$. Tax rates for a single person are in Section 1.2. How large is the tax difference $f(x_2) - f(x_1)$?

C2   What income x would leave you with \$1,000,000 after paying tax in full?

C3   For the function $f(x) = x^2 + x^3 + x^4$ compute $f(1.1), f(1.01)$, and $f(1.001)$. Subtract $f(1)$ from each of those and divide by .1, .01, and .001.

C4   Starting from $f_0 = 0$ with differences $v = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$ compute $f_{100}, f_{200}, f_{400}, f_{800}$.

Review Problems

R1   Under what condition on y_2 will the line through $(1,5)$ and $(4, y_2)$ have a positive slope?

R2   Under what condition on x_2 will the line through $(1,5)$ and $(x_2, 3)$ have a positive slope?

R3   Find the slope of the line through (x_1, y_1) and (x_2, y_2). When is the slope infinite?

R4   Explain the basic idea of *"Calculus without limits"* in Section 1.2.

R5   Explain how the average velocity $= \frac{\text{change in } f}{\text{change in } t}$ leads to an instantaneous velocity.

R6   For $f(t) = t^3$ find the average velocity between $t = 0$ and $t = h$. What is the instantaneous velocity at $t = 0$?

R7   For $f(t) = t^3$ find the average velocity between $t = 1$ and $t = 1 + h$. As $h \to 0$ find $v(1)$.

Drill Problems

D1   Find the equation of the line through $x = 0, y = 3$ with slope 5.

D2   Find the equation of the line through $(0,3)$ and $(2,7)$.

D3   Find the distance function $f(t)$ if $f(0) = 3$ and $f(2) = 7$ and the velocity is constant.

D4   If $f(0) = 6$ and $f(10) = 26$ find the average velocity between $t = 0$ and $t = 10$. *True or false:* The actual velocity is above and below that average for equal times (5 above and 5 below).

D5   For $f(t) = 7t - 5$ write down $f(t + h)$, subtract $f(t)$, and divide by h.

D6   For $f(t) = 7t^2 - 5$ write down $f(t + h)$, subtract $f(t)$, and divide by h.

D7   Find the differences v for $f = 1, 2, 4, 2, 1$. The sum of differences is _____.

D8   Find f_3 starting from $f_0 = 10$ if the differences v are $2, -6, 7$.

D9   Express $\cos 2t$ in terms of $\cos^2 t$ and then in terms of $\sin^2 t$.

Resource: Calculus Online Textbook
Gilbert Strang