PROFESSOR: And so now let's consider the spreading of infections, disease indoors.

So the McKendrick-Kermack models and various compartment models that we just described start from the assumption essentially of a well-mixed population, where there's no spatial dependence, no network dependence of connectivity of people with each other but rather sort of everyone is interacting with everyone in some sort of average sense.

So that philosophy has also been brought to bear on spreading events indoors where a little bit stronger assumption is made is that the transmission is occurring through the air and the air is well-mixed.

So not only are the people well-mixed in the sense that they can interact with each other regardless of their distance apart or anywhere placed in the room, but the reason it's a valid approximation is because the air itself is well-mixed.

And so we've already been talking about the air, the well-mixed air, and we've calculated transmission rate.

Now let's start to connect that to disease-spreading.

So one concept there would be to consider that the initial number of infected people, i0, enters a room or an indoor space with n persons, including the infectors, for a time, tau.

So we now have introduced a finite timescale, which is the time spent in the room.

So you have a room.

Some people are in it, and we're imagining an infected person comes in, and they spend a certain amount of time, and then they leave.

And during that time we'd like to ask ourselves what happened with all the other people.

How many people were infected?

OK?

So this tau kind of acts like the inverse of the removal rate, but we also will assume that gamma tau is much less than 1.

So if we think of gamma as the recovery rate or the death rate from the disease, so that's the removal quantity at the population level, then we're assuming the time spent in the room is small compared to that.

So in other words, while you're in the room, nobody is going to be recovering or dying.

We're just simply going to be transmitting the disease for a certain amount of time.

So the r is gone.

On the other hand, we can also consider an exposed compartment of the population, the group in the room.

So the exposed group is one which has essentially been exposed to the pathogen, so let's say they've inhaled a critical quantity of the virus, but they have not yet become themselves infectious.

So they are already exposed.

They are going to come down with the disease.

So they are going to be eventually symptomatic and potentially infectious, but they may not be infectious yet.

So the exposed group basically has received the disease.

So it's been transmitted to them but is not yet themselves infectious.

So that's another sort of group that we want to think about.

So another way to say it, they're not yet contagious.

So they're going to get sick, but they're not contagious yet.

So I should put here contagious is another word you could think about using.

So we've gotten rid of r, the recovered compartment, but we've added e.

So instead of the SEIR model, we have the SEI model.

Now, where we can write down the same kinds of equations as before.

ds dt is minus beta, and I'll write beta of t, si, because another complexity here is we know is that as soon as the infected person comes in if it's airborne transmission, it takes time through the fluid mechanics that we were just describing for the concentration to build up.

So the transmission rate is not constant.

We've actually already calculated that transmission rate, but it now enters this equation.

So this is now the susceptible people are getting exposed.

And so when that sort of reaction happens, if you will, the exposed population then grows, and then there's a certain rate with which exposed people can become themselves infectious to others.

And that rate we'll call alpha.

So we can remove exposed people at a rate alpha, and when they're removed they become infected people.

And so then the number of infections grows.

So this is a very closely-related model that we could solve.

And in order to couple this to airborne disease-spreading, we need to go back to our model of the airborne pathogen.

So, for example, the virions per air volume which we called c, and, in fact, I'll even write this as a dc dt with a partial derivative, which is p, which depends on the size, r, divided by-- actually this should be v-- minus lambda c of rc, so that was the model we just talked about, which we have to solve.

And then the beta of t is the breathing rate times the integral over all these sizes of c of r and t and then ci, and of course there could be masks, which might also be size dependent, dr.

So we tie it together this way.

This kind of model, which is typically run without accounting for the size dependents, but that has been done by some authors as well, generally falls under the category of Wells-Riley models.

And I mentioned here that Wells actually was a real pioneer here, already starting in the 1930s.

Did careful studies of disease transmission, including for flu and other viral diseases, and really was one of the initial proponents of the idea of infectious air, that there are particles suspended in the air which can become infectious.

He also was the one who pioneered the Wells curve, which explored evaporation versus settling.

And in 1955, he introduced this kind of a model where he was taking into account the balance of the production rate, p, and then the ventilation rate in order to describe the buildup of the infectious droplets in the air.

And then the transmission is connected in this way here.

And I should say that in Wells-Riley modeling, usually e is ignored, and so we normally just go straight from s to i.

So I'll say e is neglected, and what that really means is that the incubation rate here, this alpha, is the incubation rate.

So it's the rate at which an exposed person becomes infectious or contagious.

That alpha t is much less than 1.

So it's essentially the Wells-Riley model, it's the slow incubation limit.

But we've written down something more general here, which does also allow for the possibility of exposed people that have not yet-- that could become infectious, so we'll come back to that in just a moment.

So let's kind of summarize the results of the Wells-Riley approach.

So we'll consider the slow incubation limit, which is alpha tau much less than 1.

And so in that case, if there's slow incubation, during the time of the infected person being in the room, there is no time for new people to become infectious.

So it's really just like a fixed source.

So it's essentially saying that i is equal to i0, which is a constant.

So even though the disease is being transmitted, there is no kind of amplification effect because the people who have been exposed are not able to themselves infect others because the incubation is too slow.

And this model here, where you sort of couple these equations with the si equations basically, that was done by Gammaitoni and Nucci in 1997.

So you can see starting from Wells, we start to move towards this picture of actually coupling the dynamics of the pathogen to the transmission.

And there have been a number of models since then which have even taken into account some of these aspects of the different drop sizes and other effects that we've been discussing that were not initially considered by Wells.

So that's this.

So the first thing to do is to deal with beta of t.

So how are we going to do that?

So if I put beta t on the other side here, if I kind of divide it down, you can see that I really would like to define a new differential of a variable t hat, which is beta of t times dt.

That would be my definition of a new variable.

And so when i integrate this, it tells me that I should really switch to a new time variable, which is not just t, but it's the integral of beta dt.

So it's a time-like variable.

And once I do that I'm essentially sweeping this guy into the derivative there.

And if I also assume that i is roughly equal to i0, then what I have now is a much simpler equation here.

I go from this non-linear equation with time-dependent coefficients to something which is now linear.

And we have ds dt hat is approximately equal to beta-- sorry-- which is minus i0s.

So, again, our i is roughly constant.

Our beta got swept into the time.

So this is just kind of a simple exponential relaxation.

And the initial value is that s at time equals 0 is n minus i0.

So there are n people.

i0 of them are the initial infected ones.

And so this a pretty easy equation to solve.

It's just an exponential relaxation, which we've already seen a few times before.

And so the solution is that s is n minus i0, the initial value of s, times e to the minus i0t hat.

So it's i0, integral 0 to t of beta dt.

And then we can also write-- so that's the s, and the e is just if we have fast incubation-- or sorry, slow incubation, this term is gone, and we basically just have e as sort of just-- s and e have to add up to n basically.

And so we're left with e of t is n minus i and then 1 minus, and let me actually write this quantity up here as little q of t, where little q of t is i0 times the time integral of beta dt.

And what is this q here?

This q can be thought of as the number of infection quanta per time that are released by the infectors because their i's are infectors.

Each one of them is infecting another individual person at a rate beta, and that beta is time-dependent, so you have to integrate in time.

And so this is telling me the total number of people that would be infected by those infected people if you weren't running into all of the limits that are described by this model.

So what is happening is that if someone gets exposed, they can't be exposed again, so there's some numbers there that are changing, so you can't keep passing an infection quantum to somebody and have them get infected over and over.

So that doesn't happen.

But what this is measuring is somehow the number of times that somehow there's been an attempted infection or an expected infection if a person were susceptible.

So this q of t here is the infection quanta as defined by Wells transmitted in time t.

So up to time t, there are i0 infectors-- that number's not changing-- and the way you can think about this kind of airborne transmission is they're essentially spewing out infection quanta.

Each one of them if it hits a potentially susceptible person will infect them, but the number of susceptible people is changing.

So it's not like you keep getting more infections.

Eventually you run out of people, and you can't keep infecting them, but that's one way to think about this.

And we've already calculated this concept of infection quanta in the context of the breathing.

We've defined a quantity, which is the number of infection quanta per volume of exhaled breath.

We've also talked about the quanta emission rates for people and connected that back to the droplets.

And so here's how you see how that quantity enters into the disease-spreading models.

And in fact, we can sketch what happens with s and i here.

So basically if we start with s, for example, starts at the value n minus i0 and as you go in time it decays-- well, the decay rate is basically set by sort of the average beta.

So this is kind of like some kind of maybe if you're getting close to steady state, then beta inverse is sort of what this timescale is.

But what's actually happening is you cut it off at a certain time, tau.

So that's the time that you're in the room.

In that time, the number of susceptible people has gone down, and the number of exposed people goes up.

But then because the incubation rate is slow, the exposed people never become infectious in this model.

So the number of infectors is fixed.

And so since we have slow incubation and since the number of infectors is fixed, another way to look at this is notice this non-linear equation, which had s times i, it became i0 times s.

So it became a linear response.

And so another way to think about this limit is this is the limit of linear response.

So the Wells-Riley models are basically linear response models, which are typically not taking into account a growth in the number of infectors, which would lead to kind of an amplification of disease-spreading in a room.

And that's often justified because the time someone spends in a room is often times a lot less than the incubation time, which could be on the order of days.

People might only spend a few hours in the room.

On the other hand, there are situations such as classrooms, long-term care facilities and homes, prisons, workplaces, where people are exposed to each other for days or weeks or months actually.

They may go home in between, but there's a constant exposure.

And so you may need to worry about these other sorts of dynamics, even in an indoor setting actually.

So we'll come back to that.

Now another important concept I'd like to get here is what is the early rate of infection.

So this quantity here is q.

I wrote n minus 1 here.

That was the case.

Actually more generally this should be n minus i0 because often times we're thinking of one infector, so I kind of jumped to that, but more generally it'd be n minus i0, And this quantity here is also s0.

That's the initial number of susceptible people.

So basically you take the initial number of susceptible, and they slowly become exposed based on how many quanta they've consumed.

Now, the definition of quanta of infection often is tied to this equation.

This essentially is the Wells equation, derived in 1955.

And Wells actually defined quanta from this.

He said that one quantum infection corresponds to a probability of transmission of 63% because if you put q equals minus 1, 1 minus e to the minus 1 is 63%.

So basically he said a quantum-- so if someone asks you what is a quantum infection-- a quantum is basically 63% chance of infection.

But the problem with that thinking though is that in some other situation where there is maybe incubation going on, I might get more infected people.

So I can't just count the infected people in a room and assume that I'm getting a measure of infection quanta.

Infection quanta actually are defined by beta.

That's an important thing.

So really what is beta?

It's actually the rate of infection quantum transfer from infectious to susceptible.

Sorry.

It's si basically.

Oh actually did I-- no, that's correct, yeah.

So it's the rate of basically becoming exposed.

Excuse me.

So it's from susceptible to exposed.

So that rate basically defines beta, and that is giving you the rate of infection quanta transfer.

How many people actually get infected or exposed involves solving some set of equations like this, which might not be the same as the Wells-Riley model.

So that's an important thing to keep in mind, and we'll come back to that.

So this is basically Wells' definition of a quantum is that basically there's a 63% chance that it's going to infect somebody, but that's completely tied to this linear response exponential relaxation.

It's not really, I don't think, is the appropriate definition.

Now, another thing we can ask is, what about at early times.

And that would be the case where the expected number of quanta transferred is less than 1.

So at early times you haven't really seen a lot of infection take place.

This will be very important for us because we're going to come to safety guidelines, and in my opinion the right way to think about a safety guideline is you don't want one person to cause one or more infections.

You'd like the expectation to be that less than one person will get infected.

So this, what I'm calling early times here, in the epidemiological model is actually very relevant for, like, safety guidelines.

You don't want to deal with the case where there's rampant infection.

You just want to say if an infected person comes in the room, are they going to infect anybody else.

And you want that to be a low probability.

So this is a very relevant limit.

Now, if I take that limit here, I can expand this exponential, and I find that e of t at early times behaves like s0 times q of t if s0 is the number of susceptibles.

And if I plug in what we have here, that's n minus i0 times i0 times the integral in time of beta.

And so this is the expected number of people that will be infected by i0 infected people.

So the ratio of these two things is very important.

So if I look at e in time tau.

I should say we do this integral-- typically we want to go up to dd which is the time that the person spends in the room, and we divide by the initial number, then that's basically telling us sort of what's the reproductive number of the room.

i0 people come in, and the question is do more than i0 people come out that are infected.

So have you infected others?

So that's the ratio that you really care about.

And so I would call this the indoor reproductive number.

And in this case, if we pick i0 equal to 1, then this is n minus 1, 0 to tau beta dt.

So that's going to be an important thing, which we'll come back to later, which is the definition of kind of what makes a room actually safe.

Well, what we'd like to do is have this quantity be much less than 1 because I want to say if one infector comes in the room and everybody is currently susceptible and healthy, I want to make sure that less than one person actually gets infected And there'll be some tolerance maybe on how low I'd like that to be, but that's a very important concept.

And here you see it comes out of the slow incubation limit.

What I'd like to show you next is that the very same indoor reproductive number actually occurs for any model including the opposite limit of fast incubation.