PROFESSOR: So now, let me make the first of our technical asides, which you can skip over if you're not interested in the mathematical details or for those of you that have a higher-level, say, upper-level-undergraduate or even graduate-level understanding of transport phenomena and fluid mechanics.

I'd like to show you some of the equations that are behind the results that I've been quoting in all the lectures.

So in particular, let's derive the Wells curve.

So part of that was a theory of drop settling.

Here, I will quote a certain result, because the derivation would be a lot longer.

Actually, for that, you could refer to my online class 10.50x, which is that if you have a droplet or a particle of a radius R and it is settling under gravity-- so it has a mass m, and the gravitational force is m g, where g is the gravitational acceleration-- then there is a flow of fluid around this object.

And relative to the moving object, the flow is going the other way.

And if you solve for the viscous flow around an object being dragged through a fluid, then you arrive at the result of Stokes, which is the drag on that fluid.

So if you're falling at a velocity v, then the drag force is -- f_d is -- 6*pi times the radius of the drop times the viscosity of the fluid times the velocity of the drop.

So we're falling at a velocity v, which is the settling velocity.

And this here is the Stokes drag coefficient, which comes from solving the fluid mechanics of viscous flow around a sphere translating at a constant speed.

We can, furthermore, say that the mass of the droplet, of course, is 4*pi/3 times the density of the droplet liquid, times the radius cubed.

And so given the mass of the droplet, there's a force balance between the gravitational force m g and the drag force when the particle reaches a terminal velocity.

So if we think of this v_s as the terminal velocity where it's accelerating until there's a balance between the forces and is moving at a speed, it's given by this force balance.

And from that equation, we can solve for the settling velocity, which is energy divided by 6*pi*R.

And to use the same notation as before, I'll call this mu_a -- the air -- but generally, it's the viscosity of the ambient fluid around the particle as it's settling.

And if we plug in the value for m g, so that's 4/3*pi*rho*g* R^3 over 6*pi*R*rho_a.

And so we simplify that, we end up with (2/9)*rho*g*r^2/mu_a.

So that's the settling speed.

And this is a pretty important concept, so I'll just sketch it here.

So if we want to know what's the settling speed as a function of the radius of the drop, then you see it grows like R^2.

So it's like this.

And then to put a scale on that, if we have a particle that is 3 microns -- so that's really an aerosol particle -- then the settling speed is around 1 millimeter a second if we use the density of water and the viscosity of air for this formula.

And so that's already a fairly slow settling speed, millimeter per second.

So you can already see the particles that are in the micron range will be suspended in the air for a long period of time, as long as they don't evaporate away.

And so that is now the second part of the calculation.

Oh, and I should the finish the first part here.

What we're left with is that the settling time is L over v_s.

And that's the formula that we had before, which is 9*mu_a*L divided by 2*rho*g*r^2.

So this is our first part of the Wells curve.

So if I draw the Wells curve over here in the traditional way, where I plot on the horizontal axis the size of the particle, and in a downward axis, we draw the time, then we have a curve like this for settling.

And the reason it's drawn down, I guess maybe the feeling that as you're sort of falling down a particle a certain size, you hit this curve and that's when you've settled a distance L and fallen out of the air.

So now, let's look at evaporation, which is our second topic.

I've lost my blue [marker] -- here it is.

So these droplets are getting very small as they're evaporating, and it's happening very quickly, as we shall show in a moment.

And so a natural assumption is that the process is limited by the diffusion of water vapor away from the droplet, because essentially we have this little droplet here with a certain size R, which is now going to be varying with time.

So it has a radius R(t).

And it's really close to the surface.

There is an equilibrium concentration -- we'll call it c_w -- of water, which depends on the temperature.

So that's kind of the saturation concentration of water vapor in the air.

But then, if the water is going to evaporate more, it would create more concentration, which would then re-condense on the particle.

So in order for it to continue evaporating, that water vapor that is produced has to diffuse away.

So there's going to be a gradient of water vapor going outwards from c_w -- [it] is the concentration at position R.

And then far away, there is a sort of diffusion layer thickness, delta.

And far beyond the diffusion layer thickness, the concentration is going to approach the equilibrium concentration in the ambient air.

c at infinity is going to be c_w times the relative humidity.

So that's the ratio of the concentration of water vapor in the air to the saturation concentration, c_w, by definition.

Now, the math problem that we have to solve for this diffusion problem -- with a moving boundary in this case, though we'll assume it's pseudosteady -- is dc/dt is the diffusion coefficient of water times the Laplacian of c, so just the diffusion equation with these two boundary conditions.

Now, an interesting aspect a three-dimensional spherical diffusion is that at first the diffusion layer grows, but it very quickly reaches a steady state.

And if we assume that that diffusion time to reach this distance delta is fast, so they reach a steady state, so it's a kind of quasisteady or pseudosteady shrinking of the droplet with sort of a diffusion layer around it that's always kind of at the steady value, then it turns out that this diffusion layer is on the order of the particle size.

So as the particle shrinks, the diffusion layer also shrinks.

But it has a well-defined thickness, as opposed to diffusion in one or two dimensions, where the diffusion layer just keeps growing out to infinity.

For example, like square root of time -- you don't reach a steady state in an infinite domain.

So the bottom line of this calculation, which I will not go through right now, is that the flux of water on the surface is the area of the surface at a given moment, where the size is R, times essentially Fick's law, where the driving force, the change in concentration from the surface to the bulk, is c_w times one minus relative humidity, the diffusivity of water, and then divided by delta, the diffusion layer thickness.

And it turns out that with these coefficients here, it turns out to be exactly R.

So this is not a scaling result, but actually an exact result for pseudo steady spherical diffusion of water vapor.

So now, we have the flux on the surface.

It's uniform on the surface.

And it's [assumed] to be pseudosteady.

And so then I can write down that the change in the size of the water droplet volume, which is (4*pi/3)*R^3 is equal to minus the volume of a water molecule times the flux of water.

So that's basically my volume or mass balance of water.

So if I plug this in here, then I get dR/dt is equal to -- let's see, collecting all the terms here -- so derivative of R^3 is 3*R^2.

So the 3's cancel.

And then I have a 4*pi*R^2, which cancels this 4*pi*R^2.

So I just have dR/dt is $-v_w \cdot D_w \cdot c_w \cdot (1-RH)/R$.

If I put this R on the other side here, then I have -- I'll just continue the derivation here -- I have $R \, dR/dt$ is equal to all this stuff.

So $-v_w \cdot D_w \cdot c_w \cdot (1-RH)$.

And then this expression here can be written as 1/2 the derivative of R^2.

So what we find is that R^2 is linear in time.

And then using the boundary condition that we start out at a certain initial value, R_0, then I'm going to get the $R(t)$ is the initial value R_0 times the square root of 1 minus t over a certain evaporation time.

And that evaporation time is given here by $(R_0)^2$ times basically all these coefficients here, where I'll separate out the effective humidity, and then a bunch of other coefficients, which you can see have units of length squared over time, because R_0 is a length squared.

So it's effectively some kind of diffusivity.

And what we get from this calculation is that this effective diffusivity that goes into this expression is -- there's a factor 2 from this guy -- there's a $2 \cdot v_w \cdot D_w \cdot c_w$.

And if you plug in values for water vapor, for the saturation pressure, and the diffusivity, and the volume of water in air, then this coefficient turns out to be 1.2e-9 meters squared for second for pure water.

And that's where you get now the second part of the Wells theory, which is the evaporation, which gives you a curve looking like this.

So that there's this sort of in this theory a natural crossover between large drops, which in this case here are ones that are large enough to settle out of the fluid before they evaporate, and small drops, which evaporate.

On the other hand, for true biological fluids that appear in respiratory droplets, the evaporation is limited by solutes and salts, which stop the evaporation and, in fact, can attract even more water in some cases.

So that the evaporation part of it is not as accurate, and we tend to see that the settling part is more important to consider, given an equilibrium distribution of droplets that has been measured and is understood to come from different types of respiration.