# Bayes' Rule, Independence

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# Review of last class

- What does it mean if sets A, B, C are a partition of set D?

# Review of last class

- How do you calculate $P(A|B)$ using the formula for conditional probability?

# Review of last class

- What is the difference between P(A|B) and P(B|A)?

# Review of last class

- If B causes A, what is $P(A|B)$?

# Review of last class

- Does conditional probability require B to cause A?

# Monty Hall Problem (from last class)

- Three doors: one has $1 million behind it, the other two have nothing
- You pick one but it remains unclosed
- The host opens one door that reveals nothing (he knows which door has the prize)
- Before he opens your door (you only can pick one door), he gives you the choice of staying with your door or switching to the third door

# Monty Hall Problem

Switch or Stay?

# Bayes' Rule

- Finding P(B|A) from P(A|B)

- Using the radar/airplane example from before
  - P = plane is there
  - R = radar registers a plane

  - P(R|P) = P(radar registers | plane is present)
  - P(P|R) = P(plane is present | radar registers)

# Bayes' Rule

- How do we get P(P|R)?
- Draw the tree diagram again

# Bayes' Rule

- P(P|R) is just conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

# Bayes' Rule

- $P(P|R)$ is just conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\boxed{P(P|R) = \frac{P(P \cap R)}{P(R)}}$$

# Bayes' Rule

- Finding P(R)
- The probability that the radar registers

$$P(P|R) = \frac{P(P \cap R)}{P(R)}$$

# Bayes' Rule

- Finding $P(P \cap R)$
- The probability that the radar registers AND the plane is actually present

$$P(P|R) = \frac{P(P \cap R)}{P(R)}$$

# Bayes' Rule

- Combine the probabilities
- $P(P|R) = ?$

# Bayesian Probability

- Bayesian probability: how much you believe an event is happening based on evidence

Image courtesy of the voyager on flickr.

- The probability of a plane being present based on what a radar registers is more useful
- Example: military application

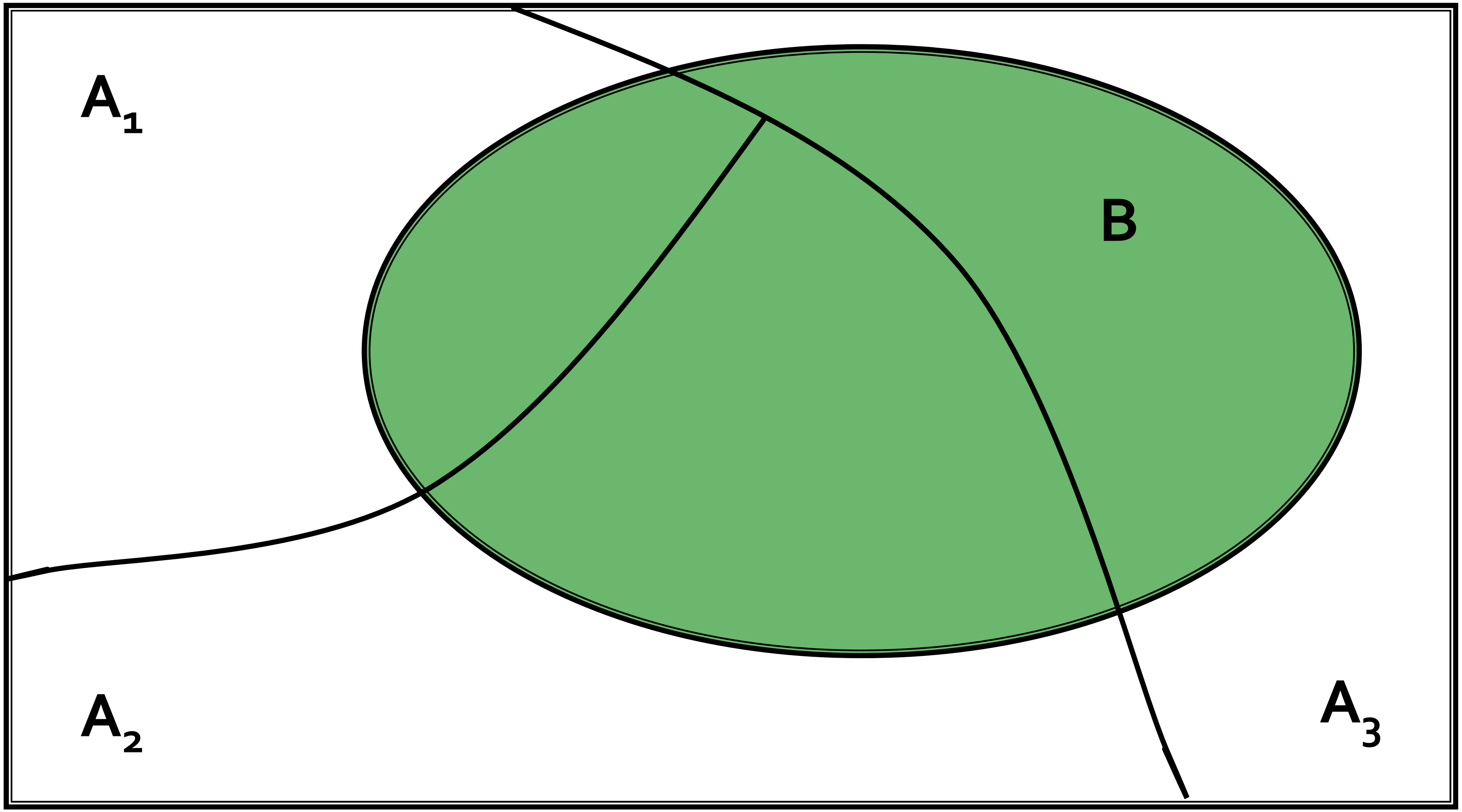
# Assumptions for Bayes' Rule

Remember: we want to find $P(B|A)$ from $P(A|B)$

- The collection of all disjoint events $A_{1\ldots n}$ are a partition of the entire sample space

- B is some event in the sample space

# Assumptions for Bayes' Rule



# Total Probability Theorem

- $P(B) = P(A_1 \cap B) + .... + P(A_n \cap B)$

$$= P(A_1)P(B|A_1) + ... + P(A_n)P(B|A_n)$$

[tree]

# Equation for Bayes' Rule

- $$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)}$$

$$= \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + \ldots + P(A_n)P(B|A_n)}$$

# Assumption for Bayes' Rule

- Sample space for the radar example?
- P = plane is present
- R = radar registers

Image courtesy of davipt.

Image courtesy of NASA.

# Problem #1: Medical Diagnosis

- A test for a rare disease is assumed to be correct 95% of the time
- If the person has the disease, the test results are positive with probability 0.95
- If the person does not have the disease, the test results are negative with probability 0.95
- A random person drawn from a certain population has probability 0.001 of getting the disease
- Given that the person tested positive, what is the probability of having the disease?

# Problem #1: Medical Diagnosis

- If you had to guess without any calculation, what would you think it was?
- P(has disease | tested positive) ?

# Problem #1: Medical Diagnosis

- Sample space? (Draw a tree?)
- What are the probabilities?

# What is Independence?

- What does it mean if $P(A|B) = P(A)$ ?

# Deriving Test for Independence

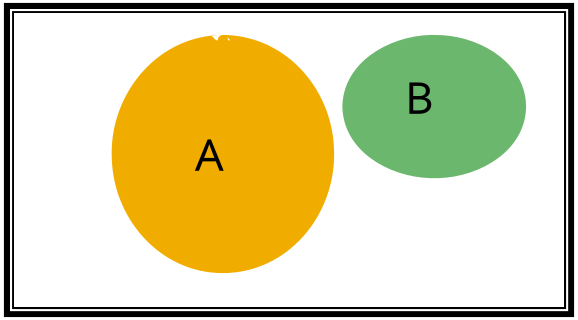
- $P(A) = P(A|B)$
- Use definition of conditional probability
  - $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

# Independent?

- Are two disjoint events independent?

- Remember the test for independence is

$$P(A \cap B) = P(A)\,P(B) \quad ?$$



# Example: Die Roll

- We throw a 4-sided die twice. Are the two successive rolls independent of each other?

- Sample space:

  { (1,1) (1,2) (1,3) (1,4)
  (2,1) (2,2) (2,3) (2,4)
  (3,1) (3,2) (3,3) (3,4)
  (4,1) (4,2) (4,3) (4,4) }

# Example: Die Roll

- Test for independence: $P(A)P(B) = P(A \cap B)$

# Problem #2

Assume a 4-sided die again, rolled twice.

Are these events independent?

A = {1st roll is 1},   B = {sum of rolls is 5}

# Problem #3

Are these events independent?

A = {maximum of the two rolls is 2}

B = {minimum of the two rolls is 2}

Can you answer it intuitively? Mathematically?

# Independence in real life

- Sometimes knowing whether events are independent is difficult
- In probability classes, problems will usually tell you to assume independence

# Conditional Independence

- Definition:

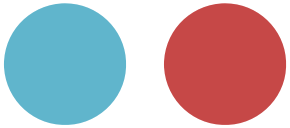
  $P(A \cap B \mid C) = P(A|C)P(B|C)$

  given C, A and B are independent

- Another way to write this:

  $P(A \mid B \cap C) = P(A|C)$

# Example: Biased Coin Toss



- We have two coins: blue and red
- We choose one of the coins at random (probability = 1/2), and toss it twice
- Tosses are independent from each other given a coin
- The blue coin lands a head 99% of the time
- The red coin lands a head 1% of the time

Events: $H_1$ = 1st toss is a head
$H_2$ = 2nd toss is a head

# Example: Biased Coin Toss

- Tosses are independent from each other GIVEN the choice of coin

conditional independence

# Problem #4: Biased Coin Toss

- What if you don't know what coin it is? Are the tosses still independent?

# Summary

- Bayes' Rule
- Total Probability Theorem
- Independence
- Conditional Independence

- Things are not always what they seem! But with these tools you can calculate the probabilities accurately