18.01 Calculus **Jason Starr**
Due by 2:00pm sharp **Fall 2005**
Friday, Oct. 21, 2005

Problem Set 4

Late homework policy. Late work will be accepted only with a medical note or for another Institute-approved reason.

Cooperation policy. You are encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding. You may not copy another student's solutions. And you should not refer to notes from a study group while writing up your solutions (if you need to refer to notes from a study group, it isn't really "your own understanding").

Part I. These problems are mostly from the textbook and reinforce the basic techniques. Occasionally the solution to a problem will be in the back of the textbook. In that case, you should work the problem first and only use the solution to check your answer.

Part II. These problems are not taken from the textbook. They are more difficult and are worth more points. When you are asked to "show" some fact, you are not expected to write a "rigorous solution" in the mathematician's sense, nor a "textbook solution". However, you should write a clear argument, using English words and complete sentences, that would convince a typical Calculus student. (Run your argument by a classmate; this is a good way to see if your argument is reasonable.) Also, for the grader's sake, try to keep your answers as short as possible (but don't leave out *important* steps).

Part I(20 points)

- **(a)** (4 points) p. 170, Section 5.2, Problem 12
- **(b)** (4 points) p. 177, Section 5.3, Problem 45
- **(c)** (4 points) p. 206, Section 6.5, Problem 1
- **(d)** (4 points) p. 212, Section 6.6, Problem 4
- **(e)** (4 points) p. 213, Section 6.6, Problem 30

Solution to (a) From $y = 1 + \frac{2}{u-1}$, we obtain $dy = -\frac{2}{(u-1)^2}du$, and from $u = \frac{v^3+6v-2}{\sqrt{v-1}}$, we obtain $du = \frac{1}{2(v-1)^{3/2}}(5v^3 - 6v^2 + 6v - 10)$, and from $v = x^4 + 5x^2 - 3$ we obtain $dv = (4x^3 + 10x)dx$.

Putting altogether, we obtain

$$dy = -\frac{2}{(u(x)-1)^2}\frac{(5v(x)^3 - 6v(x)^2 + 6v(x) - 10)}{2(v(x)-1)^{3/2}}(4x^3 + 10x)dx$$

or, equivalently,

$$dy/dx = -(5v(x)^3 - 6v(x)^2 + 6v(x) - 10)(4x^3 + 10x)/[(u(x)-1)^2(v(x)-1)^{3/2}].$$

Solution to (b) We start by rewriting $I = \int \sqrt{x^4 + x^2}dx = \int x\sqrt{1+x^2}dx$. Now let $u = 1 + x^2$, then $du = 2xdx$, and our integral becomes $I = \int u^{1/2}\frac{du}{2} = \frac{1}{3}u^{3/2} + C$. Back-substituting gives,

$$\int \sqrt{x^2 + x^4}dx = \boxed{(\sqrt{1+x^2})^3/3 + C.}$$

Solution to (c) Let $x_k = \frac{bk}{n}$, then $\Delta x_k = \frac{b}{n}$, $\bar{y}_k = \frac{b^3k^3}{n^3}$, and $S_n = A_{max} = \sum_{k=1}^n \bar{y}_k \Delta x_k = \sum_{k=1}^n \frac{b^3k^3}{n^3}\frac{b}{n} = \frac{b^4}{n^4}\sum_{k=1}^n k^3$. Using the summation formula (4) on page 195 of the textbook, we obtain

$$S_n = \frac{b^4}{n^4}\frac{n^2(n+1)^2}{4} = \frac{b^4}{4}(1+\frac{1}{n})^2 \to \boxed{b^4/4}$$

as $n \to \infty$.

Solution to (d) We first find the zeros of $4x^2 + 9y = 36$ to determine the integration limits. Thus, we let $y = 0$, and solve $4x^2 = 36$, to find the points $x = \pm 3$. Therefore the area is given by the integral

$$A = \int_{-3}^{3} ydx = \int_{-3}^{3} \frac{36 - 4x^2}{9}dx \tag{1}$$

where we used the equation $4x^2 + 9y = 36$ to express y in terms of x. The integral (1) can be evaluated easily as

$$A = \int_{-3}^{3}(4 - \frac{4}{9}x^2)dx = (4x - \frac{4}{27}x^3)|_{-3}^{3} = (4 \cdot 3 - \frac{4}{27}(27)) - (4 \cdot (-3) - \frac{4}{27}(-27)) = \boxed{16}$$

Solution to (e) Let $u = x^2 + b^2$, then $du = 2xdx$, $u(0) = b^2$ and $u(2b) = 5b^2$. Therefore

$$\int_0^{2b} \frac{xdx}{\sqrt{x^2+b^2}} = \int_{b^2}^{5b^2} u^{-1/2}\frac{du}{2} = (u^{1/2})|_{b^2}^{5b^2} = \boxed{(\sqrt{5}-1)b}$$

Part II(30 points)

Problem 1(5 points) A function $f(x)$ is defined for all $x > 0$ to be,

$$f(x) = \int_0^{\sqrt{x}} e^{-t^2}dt.$$

Using the Fundamental Theorem of Calculus, compute df/dx. Do **not** attempt to evaluate $f(x)$ directly.

Solution to Problem 1 Let $u = \sqrt{x}$, and $g(u) = \int_0^u e^{-t^2}dt = f(x)$. By the chain rule,

$$\frac{df}{dx} = \frac{dg}{du}\frac{du}{dx} = \frac{dg}{du}\frac{1}{2}x^{-1/2} \tag{2}$$

By the fundemental theorem of calculus, we have $\frac{dg}{du} = e^{-u^2} = e^{-x}$. Therefore

$$\frac{d}{dx}\int_0^{\sqrt{x}} e^{-t^2}dt = \boxed{x^{-1/2}e^{-x}/2} \tag{3}$$

Problem 2(10 points) Using the definition of the Riemann integral, upper sums, and the formulas for special sums in the textbook, compute that

$$\int_{-a}^{a}(a-x)(a+x)dx = \frac{4a^3}{3},$$

for all $a > 0$.

Solution to Problem 2 Since the function $f(x) = (a-x)(a+x) = a^2 - x^2$ is even, we can rewrite the given integral as

$$\int_{-a}^{a}(a-x)(a+x)dx = \int_{-a}^{a}(a^2-x^2)dx = 2\int_0^a(a^2-x^2)dx$$

We partition the interval $[0,a]$ uniformly, i.e. by $x_k = \frac{ka}{n}$, into n equal subintervals, each of which has length $\frac{a}{n}$. Remembering that we are required to compute the upper sums, we calculate

$$y_k = \max_{(k-1)\frac{a}{n}\le x\le k\frac{a}{n}}(a^2-x^2) = a^2 - \frac{k^2a^2}{n^2} = \frac{a^2}{n^2}(n^2-k^2).$$

Therefore

$$A_{max} = \sum_{k=1}^{n}\frac{a^2}{n^2}(n^2-k^2)\frac{a}{n} = \frac{a^3}{n^3}\sum_{k=1}^{n}(n^2-k^2)$$

$$= \frac{a^3}{n^3}(n^2n - \frac{n(n+1)(2n+1)}{6}) = a^3[1-\frac{1}{6}(2+\frac{1}{n})(1+\frac{1}{n})] \overset{n\to\infty}{\longrightarrow} a^3(1-\frac{2}{6}) = \frac{2a^3}{3}$$

Hence,

$$\int_{-a}^{a}(a-x)(a+x)dx = 2\int_0^a(a^2-x^2)dx = 2(\frac{2a^3}{4}) = \boxed{4a^3/3.}$$

Problem 3(15 points) This problem is a special case of a more general method for computing antiderivatives that will be developed systematically later in the semester.

(a)(10 points) Compute the antiderivative $\int \frac{1}{1-u^2}du$ as follows. Consider the sum $\frac{a}{u+1} + \frac{b}{u-1}$, where a and b are unspecified constants, and clear denominators. Find a choice of a and b so the sum simplifies to $\frac{1}{1-u^2}$. Now replace $\frac{1}{1-u^2}$ in the integrand by your expression $\frac{a}{u+1} + \frac{b}{u-1}$. Use $\ln(u+1)$, $\ln(u-1)$ to compute the antiderivative.

Solution to (a) Following the hint, we let

$$\frac{1}{1-u^2} = \frac{a}{1+u} + \frac{b}{1-u} = \frac{a(1-u)+b(1+u)}{1-u^2} = \frac{(-a+b)u+(a+b)}{1-u^2}$$

Therefore, we must have

$$-a + b = 0, \qquad a + b = 1$$

Solving this set of equations, we obtain $a = b = 1/2$. In summary,

$$\frac{1}{1-u^2} = \frac{1}{2}\frac{1}{1+u} + \frac{1}{2}\frac{1}{1-u} = \frac{1}{2}\frac{1}{1+u} - \frac{1}{2}\frac{1}{u-1}$$

Therefore

$$\int \frac{1}{1-u^2}du = \int \frac{1}{2}\frac{1}{1+u}du - \int \frac{1}{2}\frac{1}{u-1}du$$

$$= \frac{1}{2}\ln(|1+u|) - \frac{1}{2}\ln(|u-1|) = \frac{1}{2}\ln(\frac{|u+1|}{|u-1|}) = \boxed{\ln(\sqrt{|u+1|}/\sqrt{|u-1|}).}$$

(b)(5 points) Reduce the computation of the antiderivative $\int \frac{1}{\cos(x)}dx$ to **(a)** by multiplying numerator and denominator of the integrand by $\cos(x)$, using the trig identity $(\sin(x))^2 + (\cos(x))^2 = 1$ and substituting $u = \sin(x)$. Don't forget to back-substitute. Also, say for what range of x your antiderivative is valid ($\ln(y)$ isn't defined if y is negative!).

Solution to (b) Following the hint,

$$\int \frac{1}{\cos(x)}dx = \int \frac{\cos x}{\cos^2 x}dx = \int \frac{\cos x}{1-\sin^2 x}dx \tag{4}$$

We let $u = \sin x$, then $du = \cos x dx$, and the indefinite integral (4) becomes

$$\int \frac{du}{1-u^2} = \frac{1}{2}\ln(\frac{|u+1|}{|u-1|}) = \boxed{\ln(\sqrt{|1+\sin x|}/\sqrt{|1-\sin x|}).}$$

Because of the absolute values, the argument of the logarithm never gets negative. However, it still vanishes when $1+\sin x = 0$,(i.e. when $x = -\frac{\pi}{2} + 2\pi n$ for integers n) and still becomes singular (goes to infinity) when $1-\sin x = 0$, (i.e. when $x = \frac{\pi}{2} + 2\pi n$ for integers n). Therefore, our antiderivative function will be truly so on any interval which does not contain $\pm\frac{\pi}{2} + 2\pi n$ for any integer n. One such domain would be the open interval,

$$\boxed{(-\pi/2, \pi/2).}$$