Combinatorics: The Fine Art of Counting

Week Six Problems

In an effort to trim the fat, this week's menu has been pared to the bone. Please take a look at all four problems. The last one is especially fun.

Assume that all dice, decks, coins, etc… are standard (i.e. six-sided dice, 52 card decks, coins with heads and tails, etc...) and fair. Balls are drawn from urns without replacement unless otherwise stated.

Standard Fare

1. Which is more likely, rolling a sum of 9 with two dice, or rolling a sum of 9 with three dice? (Compute exact probabilities for both cases).

   *Rolling a 9 with two dice has probability $4/6^2$ = **1/9**, while rolling a 9 with three dice has probability $[(8\ 2) – 3]/6^3$ = **25/216** which is greater.*

   As a follow-up, what are the most likely sums when rolling two or three dice and how likely are they?

   *The most likely roll with two dice is a **7**, which has probability 6/36 = **1/6**. The most likely roll with three dice is a **10 or 11**, both of which have the same probability: $[(9\ 2)\text{-}3\text{-}6]/6^3$ = 27/216 = **1/8**.*

   If you are feeling ambitious, construct a table of all possible sums for two dice with the probabilities of each sum listed and then do the same thing for three dice (three dice are much more interesting). Keep your probabilities as unreduced fractions for easy comparison, and be sure to check that your tables add up to 1.

Sum of Two Dice

2	3	4	5	6	7	8	9	10	11	12
$\frac{1}{6^2}$	$\frac{2}{6^2}$	$\frac{3}{6^2}$	$\frac{4}{6^2}$	$\frac{5}{6^2}$	$\frac{6}{6^2}$	$\frac{5}{6^2}$	$\frac{4}{6^2}$	$\frac{3}{6^2}$	$\frac{2}{6^2}$	$\frac{1}{6^2}$

Sum of Three Dice

3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\frac{1}{6^3}$	$\frac{3}{6^3}$	$\frac{6}{6^3}$	$\frac{10}{6^3}$	$\frac{15}{6^3}$	$\frac{21}{6^3}$	$\frac{25}{6^3}$	$\frac{27}{6^3}$	$\frac{27}{6^3}$	$\frac{25}{6^3}$	$\frac{21}{6^3}$	$\frac{15}{6^3}$	$\frac{10}{6^3}$	$\frac{6}{6^3}$	$\frac{3}{6^3}$	$\frac{1}{6^3}$

2. Consider a randomly dealt hand of five cards from a standard deck. Let A be the event that the hand contains an ace. Let B be the event that the hand contains one pair (2 cards of the same rank and 3 cards of different ranks). Let C be the event that the hand contains one pair of aces (2 aces and 3 cards of different ranks). Note that the hand (A♠,4♣,4♦,K♠,J♥) is contained in both A and B but not C. Compute the following probabilities:

P(A) = 1 – P(A^c) = 1 – (48 5) / (52 5) = ***18472/54145 ~ 0.3412***

P(B) = (13 1)(4 2)(12 3)$(4\ 1)^3$ / (52 5) = ***352/833 ~ 0.4226***

P(C) = P(B) / 13 = ***352/10829 ~ 0.0325***

Let D be the event that the hand contains a pair of non-aces along with an ace and two other cards of different ranks (as in the example above).

P(D) = (12 1)(4 2)(4 1)(11 2)$(4\ 1)^2$ / (52 5) = ***1056/10829 ~ 0.0975***

P(A∪B) = P(C) + P(D) = 352/10829 + 1056/10829 = ***1408/10829*** *~* ***0.1300***

P(AUB) = P(A) + P(B) – P(A∪B) = 18472/54145 + 352/833 - 1408/10829
= ***34312/54145 ~ 0.6337***

Now suppose you know that one of the cards is an ace. Compute the following conditional probabilities:

P(B|A) = P(A∪B)/P(A) = ***880/2309 ~ 0.3811***

P(C|A) = P(A∪C)/P(A) = P(C)/P(A) = ***220/2309 ~ 0.0952***

Notice that P(B|A) < P(B), so seeing an ace decreases the chance that the hand contains a pair, but one fourth of the time that pair will be a pair of aces.

The analysis above was not particular to aces. For any particular rank, the conditional probability of getting a pair decreases. Thus knowing the rank of any one card decreases the probability of getting a pair!

This is not true if we know the exact card (e.g. the ace of spaces). The conditional probability of getting a pair given that the hand contains the ace of spades is the same as the probability of getting a pair, i.e. the events are independent.

3. An urn contains 3 red balls and *n* blue balls. The probability of drawing two blue balls is the same as the probability of drawing two balls with different colors. Determine *n*.

The probability of drawing two balls is (n 2)/(n+3 2), while the probability of drawing two balls with different colors is (3 1)(n 1)/(n+3 2). These two probabilities are equal when (n 2) = 3n which implies that n(n-7) = 0, so n must be 0 or ***7****.*

As a follow-up, replace 3 with *m* in the question above and find *n* in terms of *m*.

As above, we have (n 2) = mn which implies that n(n-2m-1) = 0, so n must be 0 or ***2m+1****.*

4. Perhaps the most famous probabilistic puzzler of all-time is what is known as the "Monty Hall" problem. This problem has baffled a surprising number of people, including more than a few mathematicians, but you should be able to solve it easily using what you have know about conditional probability.

   You are a contestant on a game show and are shown three closed doors and told that behind one of them is a brand new car while the other two have goats behind them. You are asked to pick the door you think hides the car. After you have made your choice, the game show host (Monty Hall) opens a door you did not pick and there is a goat behind it. He then asks you whether you want to change your mind and switch doors, or stick with the door you originally picked.

   Should you switch or stick? What is the probability that you will get the car?

   In view of the degree of confusion surrounding this famous problem, we will give two solutions. The first is a simple argument which uses some of the key properties of conditional probability, while the second is a more explicit analysis which gives a detailed description of the sample space and the events involved.

   Solution #1: *Without loss of generality, assume you always pick door #1 initially. If the car is behind door #2, Monty Hall must show you a goat behind door #3, and if the car is behind door #2, Monty Hall must show you a goat behind door #3. If the car is behind door #1, Monty Hall has a choice as to which door to show you, but let's assume he chooses randomly between door #2 and door #3. (Even if his choice is not random, if we average over many repeated trials of the contest, the probabilities of winning when sticking versus switching will be the same). Thus on average, half the time you are shown a goat behind door #2, and half the time you are show a goat behind door #3.*

   Let A be the event that Monty Hall shows you a goat behind door #2. P(A) = ½. Let B be the event that the car is behind door #3. P(B) = 1/3. We know that P(A|B) = 1, but what we are really interested in is P(B|A). By the law of Successive Conditioning (applied twice):

   $$P(A)P(B|A) = P(A \cap B) = P(B)P(A|B)$$

   This equivalence is also known as ***Baye's Theorem*** *and it makes it easy to convert between conditional probabilities. In this case we find that:*

   $$P(B|A) = P(B)P(A|B)/P(A) = (1/3)*1/(1/2) = 2/3$$

   Thus we should switch our choice since the car is more likely to be behind door #3 than door #1. Just to confirm it,, if we let C be the event that the car is behind door #1, P(C) is the same as P(B), but now P(A|C) = ½ instead of 1, so

$P(C|A) = P(C)P(A|C)/P(A) = (1/3)*(1/2)/(1/2) = 1/3$

as expected, since the car must be either behind door #3 or door #1 in this case. The analysis is exactly the same if we are shown a goat behind door #3 – we should always switch.

Note that the key fact is that Monty Hall has a choice when the car is behind the door we picked initially, but otherwise he does not – this is what makes the conditional probabilities different.

Solution #2: *The tricky part about this question is setting up the sample space. As noted above, the key fact is that if the car is behind the door you picked, then Monty Hall can choose either of the other two doors to open and show you a goat (but otherwise he has no choice). Let's assume he flips a coin to make this choice and if it is tails he chooses the lower numbered door. To keep the sample space uniform, we will assume he always flips a coin and just ignores the coin if he has no choice (we can assume he flips the coin ahead of time).*

To be as general as possible, our sample space will consist of all sequences of three values, the first will be our choice of door (1, 2, or 3), the second will be the door the car is behind (1, 2, or 3), and the third will be the result of the coin flip (H or T). With this sample space we can use the uniform probability measure. Our sample space contains 18 sequences, so the probability of each event will simply be the size of the event divided by 18. (There are smaller sample spaces we could use, but for the sake of clarity we will be as explicit as possible).

Let A_i be the event that we pick door i, let B_j be the event that the car is behind door j and let C_k be the event that the we are shown a goat behind door k.

We want to compute $P(B_j|A_i \cap C_k)$ where k is distinct from i and j for two cases, where j=i and where $j \neq i$. These probabilities will tell us whether to stick or switch.

Recall that $P(B_j|A_i \cap C_k) = P(B_j \cap A_i \cap C_k)/P(A_i \cap C_k)$

To be as concrete as possible, let's let i=1 and k=2 and consider the two cases where j=1 or j=3.

$A_1 = \{ (1,1,T), (1,2,T), (1,3,T), (1,1,H), (1,2,H), (1,3,H) \}$
$B_3 = \{ (1,3,T), (2,3,T), (3,3,T), (1,3,H), (2,3,H), (3,3,H) \}$
$B_1 = \{ (1,1,T), (2,1,T), (3,1,T), (1,1,H), (2,1,H), (3,1,H) \}$
Note that these events all have probability 6/18 = 1/3 as expected.

The interesting event is C_2
$C_2 = \{ (1,1,T), (1,3,T), (1,3,H), (3,1,T), (3,1,H), (3,3,H) \}$
Note how there are two sequences where door two is shown for each case where i and j are not equal (since it doesn't matter what the coin toss is), but only one sequence for each case where i=j;
We are now ready to compute the conditional probabilities.
$B_1 \cap A_1 \cap C_2 = \{ (1,1,T) \}$ *has 1 element so* $P(B_1 \cap A_1 \cap C_2) = 1/18$
$A_1 \cap C_2 = \{ (1,1,T), (1,3,T), (1,3,H) \}$, *has 3 elements so* $P(A_1 \cap C_2) = 3/18 = 1/6$
$P(B_j \cap A_i \cap C_k)/P(A_i \cap C_k) = 1/18 * 6/1 = \mathbf{1/3}$

So the probability we will get the car if we stick is 1/3. It must be the case that the probability of getting the car is 2/3 if we switch, but let's compute it explicitly just to see how the conditional probabilities come out.

$B_3 \cap A_1 \cap C_2$ = { (1,3,T), (1,3,H) } has 2 elements so $P(B_1 \cap A_1 \cap C_2)$ = 2/18
*$P(B_j \cap A_i \cap C_k)/P(A_i \cap C_k)$ = 2/18 * 6/1 = **2/3** as expected.*

The analysis above works for any i and j. ***Thus we should always switch, and doing so will result in winning the car with probability 2/3.***

Note that the analysis above can be generalized to more than 3 doors (still one car, but lots of goats). In this scenario Monty Hall could open more than one door with a goat behind it. The more doors he opens the greater the probability of winning the car if you switch. Considering the case where there are 100 doors and after you pick door 1 Monty Hall opens every other door except one (say door number 82) is a useful way to better understand why switching is the right strategy.

If you want to try playing this game, visit the following web-site:

http://math.ucsd.edu/~crypto/Monty/monty.html

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