Stochastic Calculus and Option Pricing

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Outline

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Brownian Motion

- Consider a random walk $x_{t+n\Delta t}$ with equally likely increments of $\pm\sqrt{\Delta t}$.
- Let the time step of the random walk shrink to zero: $\Delta t \to 0$.
- The limit is a continuous-time process called *Brownian motion*, which we denote Z_t, or $Z(t)$.
- We always set $Z_0 = 0$.
- Brownian motion is a basic building block of continuous-time models.

Properties of Brownian Motion

- Brownian motion has independent increments: if $t < t' < t''$, then $Z_{t'} - Z_t$ is independent of $Z_{t''} - Z_{t'}$.
- Increments of the Brownian motion have normal distribution with zero mean, and variance equal to the time interval between the observation points

$$Z_{t'} - Z_t \sim \mathcal{N}(0, t' - t)$$

  Thus, for example,

$$\mathrm{E}_t[Z_{t'} - Z_t] = 0$$

  Intuition: Central Limit Theorem applied to the random walk.
- Trajectories of the Brownian motion are continuous.
- Trajectories of the Brownian motion are nowhere differentiable, therefore standard calculus rules do not apply.

Ito Integral

- Ito integral, also called the stochastic integral (with respect to the Brownian motion) is an object

$$\int_0^t \sigma_u \, dZ_u$$

  where σ_u is a stochastic process.

- Important: σ_u can depend on the past history of Z_u, but it cannot depend on the future. σ_u is called adapted to the history of the Brownian motion.
- Consider discrete-time approximations

$$\sum_{i=1}^{N} \sigma_{(i-1)\Delta t}(Z_{i\Delta t} - Z_{(i-1)\Delta t}), \quad \Delta t = \frac{t}{N}$$

  and then take the limit of $N \to \infty$ (the limit must be taken in the mean-squared-error sense).

- The limit is well defined, and is called the Ito integral.

Properties of Itô Integral

- Itô integral is linear

$$\int_0^t (a_u + c \times b_u)\, dZ_u = \int_0^t a_u\, dZ_u + c \int_0^t b_u\, dZ_u$$

- $X_t = \int_0^t \sigma_u\, dZ_u$, is continuous as a function of time.
- Increments of X_t have conditional mean of zero (under some technical restrictions on σ_u):

$$\mathrm{E}_t[X_{t'} - X_t] = 0, \quad t' > t$$

  Note: a sufficient condition for $\mathrm{E}_t\left[\int_t^T \sigma_u\, dZ_u\right] = 0$ is $\mathrm{E}_0\left[\int_0^T \sigma_u^2\, du\right] < \infty$.
- Increments of X_t are uncorrelated over time.
- If σ_t is a deterministic function of time and $\int_0^t \sigma_u^2\, du < \infty$, then X_t is normally distributed with mean zero, and variance

$$\mathrm{E}_0[X_t^2] = \int_0^t \sigma_u^2\, du$$

Itô Processes

- An Itô process is a continuous-time stochastic process X_t, or $X(t)$, of the form

$$\int_0^t \mu_u \, du + \int_0^t \sigma_u \, dZ_u$$

- μ_u is called the instantaneous drift of X_t at time u, and σ_u is called the instantaneous volatility, or the diffusion coefficient.
- $\mu_t \, dt$ captures the expected change of X_t between t and $t + dt$.
- $\sigma_t \, dZ_t$ captures the unexpected (stochastic) component of the change of X_t between t and $t + dt$.
- Conditional mean and variance:

$$\mathrm{E}_t(X_{t+dt} - X_t) = \mu_t \, dt + o(dt), \quad \mathrm{E}_t\left[(X_{t+dt} - X_t)^2\right] = \sigma_t^2 \, dt + o(dt)$$

Outline

Quadratic Variation

- Consider a time discretization

$$0 = t_1 < t_2 < \ldots < t_N = T, \qquad \max_{n=1,\ldots,N-1} |t_{n+1} - t_n| < \overline{\Delta}$$

- Quadratic variation of an Itô process $X(t)$ between 0 and T is defined as

$$[X]_T = \lim_{\overline{\Delta} \to 0} \sum_{n=1}^{N-1} |X(t_{n+1}) - X(t_n)|^2$$

- For the Brownian motion, quadratic variation is deterministic:

$$[Z]_T = T$$

  To see the intuition, consider the random-walk approximation to the Brownian motion: each increment equals $\sqrt{t_{n+1} - t_n}$ in absolute value.

Quadratic Variation

- Quadratic variation of an Itô process

$$X_t = \int_0^t \mu_u \, du + \int_0^t \sigma_u \, dZ_u$$

is given by

$$[X]_T = \int_0^T \sigma_t^2 \, dt$$

- Heuristically, the quadratic variation formula states that

$$(dZ_t)^2 = dt, \quad dt \, dZ_t = o(dt), \quad (dt)^2 = o(dt)$$

- Random walk intuition:

$$|dZ_t| = \sqrt{dt}, \quad |dt \, dZ_t| = (dt)^{3/2} = o(dt), \quad dZ_t^2 = dt$$

- Conditional variance of the Itô process can be estimated by approximating its quadratic variation with a discrete sum. This is the basis for variance estimation using high-frequency data.

Itô's Lemma

- Itô's Lemma states that if X_t is an Itô process,

$$X_t = \int_0^t \mu_u \, du + \int_0^t \sigma_u \, dZ_u$$

then so is $f(t, X_t)$, where f is a sufficiently smooth function, and

$$df(t, X_t) = \left(\frac{\partial f(t, X_t)}{\partial t} + \frac{\partial f(t, X_t)}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2} \sigma_t^2 \right) dt + \frac{\partial f(t, X_t)}{\partial X_t} \sigma_t \, dZ_t$$

- Itô's Lemma is, heuristically, a second-order Taylor expansion in t and X_t, using the rule that

$$(dZ_t)^2 = dt, \quad dt \, dZ_t = o(dt), \quad (dt)^2 = o(dt)$$

Itô's Lemma

- Using the Taylor expansion,

$$
\begin{aligned}
df(t, X_t) \approx & \frac{\partial f(t, X_t)}{\partial t}\, dt + \frac{\partial f(t, X_t)}{\partial X_t}\, dX_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial t^2}\, dt^2 \\
& + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2}\, (dX_t)^2 + \frac{\partial^2 f(t, X_t)}{\partial t \partial X_t} dt\, dX_t \\
= & \frac{\partial f(t, X_t)}{\partial t}\, dt + \frac{\partial f(t, X_t)}{\partial X_t} \mu_t\, dt + \frac{\partial f(t, X_t)}{\partial X_t} \sigma_t\, dZ_t \\
& + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2} \sigma_t^2\, dt + o(dt)
\end{aligned}
$$

- Short-hand notation

$$
df(t, X_t) = \frac{\partial f(t, X_t)}{\partial t}\, dt + \frac{\partial f(t, X_t)}{\partial X_t}\, dX_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2}\, (dX_t)^2
$$

Itô's Lemma

Example

- Let $X_t = \exp(at + bZ_t)$.
- We can write $X_t = f(t, Z_t)$, where

$$f(t, Z_t) = \exp(at + bZ_t)$$

- Using

$$\frac{\partial f(t, Z_t)}{\partial t} = af(t, Z_t), \quad \frac{\partial f(t, Z_t)}{\partial Z_t} = bf(t, Z_t), \quad \frac{\partial^2 f(t, Z_t)}{\partial Z_t^2} = b^2 f(t, Z_t)$$

  Itô's Lemma implies

$$dX_t = \left(a + \frac{b^2}{2}\right) X_t \, dt + bX_t \, dZ_t$$

- Expected growth rate of X_t is $a + b^2/2$.

Outline

The Black-Scholes Model of the Market

- Consider the market with a constant risk-free interest rate r and a single risky asset, the stock.
- Assume the stock does not pay dividends and the price process of the stock is given by

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z_t\right)$$

- Because Brownian motion is normally distributed, using $\mathrm{E}_0[\exp(Z_t)] = \exp(t/2)$, find

$$\mathrm{E}_0[S_t] = S_0 \exp(\mu t)$$

- Using Itô's Lemma (check)

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dZ_t$$

- μ is the expected continuously compounded stock return, σ is the volatility of stock returns.
- Stock returns have constant volatility.

Dynamic Trading

- Consider a trading strategy with continuous rebalancing.
- At each point in time, hold θ_t shares of stocks in the portfolio.
- Let the portfolio value be W_t. Then $W_t - \theta_t S_t$ dollars are invested in the short-term risk-free bond.
- Portfolio is self-financing: no exogenous incoming or outgoing cash flows.
- Portfolio value changes according to

$$dW_t = \theta_t \, dS_t + (W_t - \theta_t S_t) r \, dt$$

- Discrete-time analogy

$$W_{t+\Delta t} - W_t = \theta_t (S_{t+\Delta t} - S_t) + (W_t - \theta_t S_t) \left(\frac{B_{t+\Delta t}}{B_t} - 1 \right)$$

Option Replication

- Consider a European option with the payoff $H(S_T)$.
- We will construct a self-financing portfolio replicating the payoff of the option.
- Look for the portfolio such that

$$W_t = f(t, S_t)$$

  for some function $f(t, S_t)$.
- By Law of One Price, $f(t, S_t)$ must be the price of the option at time t, being the cost of a trading strategy with an identical payoff.
- Note that the self-financing condition is important for the above argument: we do not want the portfolio to produce intermediate cash flows.

Option Replication

- Apply Itô's Lemma to portfolio value, $W_t = f(t, S_t)$:

$$dW_t = \theta_t \, dS_t + (W_t - \theta_t S_t) r \, dt = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} (dS_t)^2$$

  where $(dS_t)^2 = \sigma^2 S_t^2 \, dt$
- The above equality holds at all times if

$$\theta_t = \frac{\partial f(t, S_t)}{\partial S_t}, \quad \frac{\partial f(t, S_t)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(t, S_t)}{\partial S_t^2} \sigma^2 S_t^2 - r \left(f(t, S_t) - \frac{\partial f}{\partial S_t} S_t \right) = 0$$

- If we can find the solution $f(t, S)$ to the PDE

$$-r \, f(t, S) + \frac{\partial f(t, S)}{\partial t} + rS \frac{\partial f(t, S)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f(t, S)}{\partial S^2} = 0$$

  with the boundary condition $f(T, S) = H(S)$, then the portfolio with

$$W_0 = f(0, S_0), \quad \theta_t = \frac{\partial f(t, S_t)}{\partial S_t}$$

  replicates the option!

Black-Scholes Option Price

- We conclude that the option price can be computed as a solution of the Black-Scholes PDE

$$-rf + \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = 0$$

- If the option is a European call with strike K, the PDE can be solved in closed form, yielding the Black-Scholes formula:

$$C(t, S_t) = S_t N(z_1) - \exp(-r(T-t))KN(z_2),$$

where $N(.)$ is the cumulative distribution function of the standard normal distribution,

$$z_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$z_2 = z_1 - \sigma\sqrt{T-t}.$$

- Note that μ does not enter the PDE or the B-S formula. This is intuitive from the perspective of risk-neutral pricing. Discuss later.

Black-Scholes Option Replication

- The replicating strategy requires holding

$$\theta_t = \frac{\partial f(t, S_t)}{\partial S_t}$$

  stock shares in the portfolio. θ_t is called the option's delta.
- It is possible to replicate any option in the Black-Scholes setting because
  - Price of the stock S_t is driven by a Brownian motion;
  - Rebalancing of the replicating portfolio is continuous;
  - There is a single Brownian motion affecting the payoff of the option and the price of the stock. More on this later, when we cover multivariate Itô processes.

Single-Factor Term Structure Model

- Consider a model of the term structure of default-free bond yields.
- Assume that the short-term interest rate follows

$$dr_t = \alpha(r_t)\, dt + \beta(r_t)\, dZ_t$$

- Let $P(t, \tau)$ denote the time-t price of a discount bond with unit face value maturing at time τ.
- We want to construct an arbitrage-free model capturing, simultaneously, the dynamics of bond prices of many different maturities.

Single-Factor Term Structure Model

- Assume that bond prices can be expressed as a function of the short rate only (single-factor structure)

$$P(t,\tau) = f(t, r_t, \tau)$$

- Itô formula implies that

$$dP(t,\tau) = \underbrace{\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\alpha(r_t) + \frac{1}{2}\frac{\partial^2 f}{\partial r^2}\beta^2(r_t)\right)}_{\mu_t^\tau} dt + \underbrace{\frac{\partial f}{\partial r}\beta(r_t)}_{\sigma_t^\tau} dZ_t$$

- Consider a self-financing portfolio which invests $P(t,\tau)/\sigma_t^\tau$ dollars in the bond maturing at τ, and $-P(t,\tau')/\sigma_t^{\tau'}$ dollars in the bond maturing at τ'.
- Self-financing requires that the investment in the risk-free short-term bond is

$$W_t - \frac{P(t,\tau)}{\sigma_t^\tau} + \frac{P(t,\tau')}{\sigma_t^{\tau'}}$$

Single-Factor Term Structure Model

- Portfolio value evolves according to

$$dW_t = \left[\left(W_t - \frac{P(t,\tau)}{\sigma_t^\tau} + \frac{P(t,\tau')}{\sigma_t^{\tau'}} \right) r_t + \frac{\mu_t^\tau}{\sigma_t^\tau} - \frac{\mu_t^{\tau'}}{\sigma_t^{\tau'}} \right] dt + \left[\frac{\sigma_t^\tau}{\sigma_t^\tau} - \frac{\sigma_t^{\tau'}}{\sigma_t^{\tau'}} \right] dZ_t$$

$$= \left[\left(W_t - \frac{P(t,\tau)}{\sigma_t^\tau} + \frac{P(t,\tau')}{\sigma_t^{\tau'}} \right) r_t + \frac{\mu_t^\tau}{\sigma_t^\tau} - \frac{\mu_t^{\tau'}}{\sigma_t^{\tau'}} \right] dt$$

- Portfolio value changes are instantaneously risk-free.
- To avoid arbitrage, the portfolio value must grow at the risk-free rate:

$$\left(W_t - \frac{P(t,\tau)}{\sigma_t^\tau} + \frac{P(t,\tau')}{\sigma_t^{\tau'}} \right) r_t + \frac{\mu_t^\tau}{\sigma_t^\tau} - \frac{\mu_t^{\tau'}}{\sigma_t^{\tau'}} = W_t r_t$$

Single-Factor Term Structure Model

- We conclude that

$$\frac{\mu_t^\tau - r_t P(t,\tau)}{\sigma_t^\tau} = \frac{\mu_t^{\tau'} - r_t P(t,\tau')}{\sigma_t^{\tau'}}, \quad \text{for any } \tau \text{ and } \tau'$$

- Assume that, for some τ',

$$\frac{\mu_t^{\tau'} - r_t P(t,\tau')}{\sigma_t^{\tau'}} = \eta(t, r_t)$$

- Then, for all bonds, must have

$$\mu_t^\tau - r_t P(t,\tau) = \eta(t, r_t)\sigma_t^\tau$$

- Recall the definition of μ_t^τ, σ_t^τ to derive the pricing PDE on $P(t,\tau) = f(t, r_t, \tau)$

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\alpha(r) + \frac{1}{2}\frac{\partial^2 f}{\partial r^2}\beta^2(r) - rf = \eta(t,r)\beta(r)\frac{\partial f}{\partial r}, \quad f(\tau, r, \tau) = 1$$

- The solution, indeed, has the form $f(t, r, \tau)$.

Single-Factor Term Structure Model

- What if
$$\frac{\mu_t^\tau - r_t P(t,\tau)}{\sigma_t^\tau} \neq \eta(t, r_t)$$
for any function η, i.e., the LHS depends on something other than r_t and t? Then the term structure will not have a single-factor form.
- We have seen that, for each choice of $\eta(t, r_t)$, absence of arbitrage implies that bond prices must satisfy the pricing PDE.
- The reverse is true: if bond prices satisfy the pricing PDE (with well-behaved $\eta(t, r_t)$), there is no arbitrage (show later, using risk-neutral pricing).
- The choice of $\eta(t, r_t)$ determines the joint arbitrage-free dynamics of bond prices (yields).
- $\eta(t, r_t)$ is the price of interest rate risk.

Outline

Multiple Brownian motions

- Consider two independent Brownian motions Z_t^1 and Z_t^2. Construct a third process

$$X_t = \rho Z_t^1 + \sqrt{1-\rho^2} Z_t^2$$

- X_t is also a Brownian motion:
  - X_t has IID normal increments;
  - X_t is continuous.
- X_t and Z_t^1 are correlated:

$$\mathrm{E}_0\left[X_t Z_t^1\right] = \rho t$$

- Correlated Brownian motions can be constructed from uncorrelated ones, just like with normal random variables.
- Cross-variation

$$[Z_t^1, Z_t^2]_T = \lim_{\Delta \to 0} \sum_{n=1}^{N-1} \left(Z^1(t_{n+1}) - Z^1(t_n)\right) \times \left(Z^2(t_{n+1}) - Z^2(t_n)\right) = 0$$

- Short-hand rule

$$dZ_t^1 \, dZ_t^2 = 0 \;\Rightarrow\; dZ_t^1 \, dX_t = \rho \, dt$$

Multivariate Itô Processes

- A multivariate Itô process is a vector process with each coordinate driven by an Itô process.
- Consider a pair of processes

$$
\begin{aligned}
dX_t &= \mu_t^X \, dt + \sigma_t^X \, dZ_t^X, \\
dY_t &= \mu_t^Y \, dt + \sigma_t^Y \, dZ_t^Y, \\
dZ_t^X \, dZ_t^Y &= \rho_t \, dt
\end{aligned}
$$

- Itô's formula can be extended to multiple process as follows:

$$
\begin{aligned}
df(t, X_t, Y_t) = & \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial X_t} \, dX_t + \frac{\partial f}{\partial Y_t} \, dY_t + \\
& \frac{1}{2} \frac{\partial^2 f}{\partial X_t^2} (dX_t)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial Y_t^2} (dY_t)^2 + \frac{\partial^2 f}{\partial X_t \partial Y_t} \, dX_t \, dY_t
\end{aligned}
$$

Example of Itô's formula

- Consider two asset price processes, X_t and Y_t, both given by Itô processes

$$dX_t = \mu_t^X \, dt + \sigma_t^X \, dZ_t^X$$
$$dY_t = \mu_t^Y \, dt + \sigma_t^Y \, dZ_t^Y$$
$$dZ_t^X \, dZ_t^Y = 0$$

- Using Itô's formula, we can derive the process for the ratio $f_t = X_t/Y_t$ (use $f(X, Y) = X/Y$):

$$\frac{df_t}{f_t} = \frac{dX_t}{X_t} - \frac{dY_t}{Y_t} - \frac{dX_t}{X_t}\frac{dY_t}{Y_t} + \left(\frac{dY_t}{Y_t}\right)^2$$
$$= \left(\frac{\mu_t^X}{X_t} - \frac{\mu_t^Y}{Y_t} + \frac{(\sigma_t^Y)^2}{Y_t^2}\right) dt + \frac{\sigma_t^X}{X_t} \, dZ_t^X - \frac{\sigma_t^Y}{Y_t} \, dZ_t^Y$$

Example of Itô's formula

- We find that the expected growth rate of the ratio X_t/Y_t is

$$\left(\frac{\mu_t^X}{X_t} - \frac{\mu_t^Y}{Y_t} + \frac{\left(\sigma_t^Y\right)^2}{Y_t^2}\right)$$

- Assume that $\mu_t^X = \mu_t^Y$. Then,

$$\mathrm{E}_t\left(\frac{df_t}{f_t}\right) = \frac{\left(\sigma_t^Y\right)^2}{Y_t^2}\,dt$$

- Repeating the same calculation for the inverse ratio, $h_t = Y_t/X_t$, we find

$$\mathrm{E}_t\left(\frac{dh_t}{h_t}\right) = \frac{\left(\sigma_t^X\right)^2}{X_t^2}\,dt$$

- It is possible for both the ratio X_t/Y_t and its inverse Y_t/X_t to be expected to grow at the same time. Application to FX.

Outline

Stochastic Differential Equations

- Example: Heston's stochastic volatility model

$$\frac{dS_t}{S_t} = \mu\, dt + \sqrt{v_t}\, dZ_t^S$$
$$dv_t = -\kappa(v_t - \overline{v})\, dt + \gamma\sqrt{v_t}\, dZ_t^v$$
$$dZ_t^S\, dZ_t^v = \rho\, dt$$

Conditional variance v_t is described by a Stochastic Differential Equation.

Definition (SDE)

The Itô process X_t satisfies a stochastic differential equation

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dZ_t$$

with an initial condition X_0 if it satisfies

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \sigma(s, X_s)\, dZ_s.$$

Existence of Solutions of SDEs

- Assume that for some $C, D > 0$

$$|\mu(t,X)| + |\sigma(t,X)| \leqslant C(1+|X|)$$

  and

$$|\mu(t,X) - \mu(t,Y)| + |\sigma(t,X) - \sigma(t,Y)| \leqslant D|X-Y|$$

  for any X and Y (Lipschitz property).

- Then, the SDE

$$dX_t = \mu(t,X_t)\,dt + \sigma(t,X_t)\,dZ_t, \quad X_0 = x,$$

  has a unique continuous solution X_t.

Common SDEs

Arithmetic Brownian Motion

- The solution of the SDE

$$dX_t = \mu\, dt + \sigma\, dZ_t$$

  is given by

$$X_t = X_0 + \mu t + \sigma Z_t.$$

  The process X_t is called an arithmetic Brownian motion, or Brownian motion with a drift.
- Guess and verify.
- We typically reduce an SDE to a few common cases with explicit solutions.

Common SDEs

Geometric Brownian Motion

- Consider the SDE

$$dX_t = \mu X_t \, dt + \sigma X_t \, dZ_t$$

- Define the process

$$Y_t = \ln(X_t).$$

By Itô's Lemma,

$$dY_t = \frac{1}{X_t}\mu X_t \, dt + \frac{1}{X_t}\sigma X_t \, dZ_t + \frac{1}{2}\left(-\frac{1}{X_t^2}\right)\sigma^2 X_t^2 \, dt = (\mu - \sigma^2/2) \, dt + \sigma \, dZ_t.$$

- Y_t is an arithmetic Brownian motion, given in the previous example, and

$$Y_t = Y_0 + (\mu - \sigma^2/2)t + \sigma Z_t.$$

- Then

$$X_t = e^{Y_t} = X_0 \exp\left((\mu - \sigma^2/2)t + \sigma Z_t\right)$$

Common SDEs

Ornstein-Uhlenbeck process

- The mean-reverting Ornstein-Uhlenbeck process is the solution X_t to the stochastic differential equation

$$dX_t = (\overline{X} - X_t)\, dt + \sigma\, dZ_t$$

- We solve this equation using e^t as an integrating factor.
- Setting $Y_t = e^t$ and using Itô's lemma for the function $f(X, Y) = X\, Y$, we find

$$d(e^t X_t) = e^t(\overline{X}\, dt + \sigma\, dZ_t).$$

Integrating this between 0 and t, we find

$$e^t X_t - X_0 = \int_0^t e^s \overline{X} ds + \int_0^t e^s \sigma dZ_s,$$

i.e.,

$$X_t = e^{-t} X_0 + (1 - e^{-t})\overline{X} + \sigma \int_0^t e^{s-t} dZ_s.$$

Option Replication in the Heston Model

- Assume the Heston stochastic-volatility model for the stock.
- Attempt to replicate the option payoff with the stock and the risk-free bond.
- Can we find a trading strategy that would guarantee perfect replication?
- It is possible to replicate an option using a bond, a stock, and another option.

Recap of APT

- Recall the logic of APT.
- Suppose we have N assets with two-factor structure in their returns:

$$R_t^i = a_i + b_i^1 F_t^1 + b_i^2 F_t^2$$

- Interest rate is r.
- While not stated explicitly, all factor loadings may be stochastic.
- Unlike the general version of the APT, we assume that returns have no idiosyncratic component.
- At time t, consider any portfolio with fraction θ_i in each asset i that has zero exposure to both factors:

$$b_1^1 \theta_1 + b_1^2 \theta_2 + \ldots + b_1^N \theta_N = 0$$
$$b_2^1 \theta_1 + b_2^2 \theta_2 + \ldots + b_2^N \theta_N = 0$$

- This portfolio must have zero expected excess return to avoid arbitrage

$$\theta_1 \left(\mathrm{E}_t[R_t^1] - r \right) + \theta_2 \left(\mathrm{E}_t[R_t^2] - r \right) + \ldots + \theta_N \left(\mathrm{E}_t[R_t^N] - r \right) = 0$$

Recap of APT

- To avoid arbitrage, any portfolio satisfying

$$b_1^1\theta_1 + b_1^2\theta_2 + \ldots + b_1^N\theta_N = 0$$
$$b_2^1\theta_1 + b_2^2\theta_2 + \ldots + b_2^N\theta_N = 0$$

must satisfy

$$\theta_1\left(\mathrm{E}_t[R_t^1] - r\right) + \theta_2\left(\mathrm{E}_t[R_t^2] - r\right) + \ldots + \theta_N\left(\mathrm{E}_t[R_t^N] - r\right) = 0$$

- Restating this in vector form, any vector orthogonal to

$$(b_1^1, b_1^2, \ldots, b_1^N) \text{ and } (b_2^1, b_2^2, \ldots, b_2^N)$$

must be orthogonal to $(\mathrm{E}_t[R_t^1] - r, \mathrm{E}_t[R_t^2] - r, \ldots, \mathrm{E}_t[R_t^N] - r)$.

- Conclude that the third vector is spanned by the first two: there exist constants (prices of risk) $(\lambda_t^1, \lambda_t^2)$ such that

$$\mathrm{E}_t[R_t^i] - r = \lambda_t^1 b_1^i + \lambda_t^2 b_2^i, \quad i = 1, \ldots, N$$

Option Pricing in the Heston Model

- Suppose there are N derivatives with prices given by

$$f^i(t, S_t, v_t)$$

where the first option is the stock itself: $f^1(t, S_t, v_t) = S_t$.

- Using Ito's lemma, their prices satisfy

$$df^i(t, S_t, v_t) = a_t^i dt + \frac{\partial f^i(t, S_t, v_t)}{\partial S_t} dS_t + \frac{\partial f^i(t, S_t, v_t)}{\partial v_t} dv_t$$

- Compare the above to our APT argument
- Conclude that there exist λ_t^S and λ_t^v such that

$$\mathrm{E}\left[df^i(t, S_t, v_t) - rf^i(t, S_t, v_t)\, dt\right] = \frac{\partial f^i(t, S_t, v_t)}{\partial S_t}\lambda_t^S\, dt + \frac{\partial f^i(t, S_t, v_t)}{\partial v_t}\lambda_t^v\, dt$$

- We work with price changes instead of returns, as we did in the APT, because some of the derivatives may have zero price.

Option Pricing in the Heston Model

- The APT pricing equation, applied to the stock, implies that

$$\mathrm{E}\left[dS_t - rS_t\,dt\right] = (\mu - r)S_t\,dt = \lambda_t^S\,dt$$

- λ_t^v is the price of volatility risk, which determines the risk premium on any investment with exposure to dv_t.
- Writing out the pricing equation explicitly, with the Ito's lemma providing an expression for $\mathrm{E}\left[df^i(t, S_t, v_t)\right]$,

$$\frac{\partial f^i}{\partial t} + \frac{\partial f^i}{\partial S}\mu S + \frac{\partial f^i}{\partial v}(-\kappa)(v - \overline{v}) +$$

$$\frac{1}{2}\frac{\partial^2 f^i}{\partial S^2} vS^2 + \frac{1}{2}\frac{\partial^2 f^i}{\partial v^2}\gamma^2 v + \frac{\partial^2 f^i}{\partial S \partial v}\rho\gamma S v - rf^i = \frac{\partial f^i}{\partial S}(\mu - r)S + \frac{\partial f^i}{\partial v}\lambda_t^v$$

- As long as we assume that the price of volatility risk is of the form

$$\lambda_t^v = \lambda^v(t, S_t, v_t)$$

the assumed functional form for option prices is justified and we obtain an arbitrage-free option pricing model.

Numerical Solution of SDEs

- Except for a few special cases, SDEs do not have explicit solutions.
- The most basic and common method of approximating solutions of SDEs numerically is using the first-order Euler scheme.
- Use the grid $t_i = i\Delta$.

$$\widehat{X}_{i+1} = \widehat{X}_i + \mu(t_i, \widehat{X}_i)\,\Delta + \sigma(t_i, \widehat{X}_i)\,\sqrt{\Delta}\widetilde{\varepsilon}_i,$$

  where $\widetilde{\varepsilon}_i$ are IID $\mathcal{N}(0,1)$ random variables.
- Using a binomial distribution for $\widetilde{\varepsilon}_i$, with equal probabilities of ± 1, is also a valid procedure for approximating the distribution of X_t

Outline

Moments of Diffusion Processes

- Often need to compute conditional moments of diffusion processes:
  - Expected returns and variances of returns on financial assets over finite time intervals;
  - Use the method of moments to estimate a diffusion process from discretely sampled data;
  - Compute prices of derivatives.
- One approach is to reduce the problem to a PDE, which can sometimes be solved analytically.

Kolmogorov Backward Equation

- Diffusion process X_t with coefficients $\mu(t, X)$ and $\sigma(t, X)$.
- Objective: compute a conditional expectation

$$f(t, X) = E[g(X_T)|X_t = X]$$

- Suppose $f(t, X)$ is a smooth function of t and X. By the law of iterated expectations,

$$f(t, X_t) = E_t[f(t + dt, X_{t+dt})] \quad \Rightarrow \quad E_t[df(t, X_t)] = 0$$

- Using Ito's Lemma,

$$E_t[df(t, X_t)] = \left(\frac{\partial f(t, X)}{\partial t} + \mu(t, X) \frac{\partial f(t, X)}{\partial X} + \frac{1}{2} \sigma(t, X)^2 \frac{\partial^2 f(t, X)}{\partial X^2} \right) dt = 0$$

- Kolmogorov backward equation

$$\frac{\partial f(t, X)}{\partial t} + \mu(t, X) \frac{\partial f(t, X)}{\partial X} + \frac{1}{2} \sigma(t, X)^2 \frac{\partial^2 f(t, X)}{\partial X^2} = 0,$$

with boundary condition

$$f(T, X) = g(X)$$

Example: Square-Root Diffusion

- Consider a popular diffusion process used to model interest rates and stochastic volatility

$$dX_t = -\kappa(X_t - \overline{X})\, dt + \sigma\sqrt{X_t}\, dZ_t$$

- We want to compute the conditional moments of this process, to be used as a part of GMM estimation.
- Compute the second non-central moment

$$f(t, X) = \mathrm{E}(X_T^2 | X_t = X)$$

- Using Kolmogorov backward equation,

$$\frac{\partial f(t, X)}{\partial t} - \kappa(X - \overline{X})\frac{\partial f(t, X)}{\partial X} + \frac{1}{2}\sigma^2 X \frac{\partial^2 f(t, X)}{\partial X^2} = 0,$$

with boundary condition

$$f(T, X) = X^2$$

Example: Square-Root Diffusion

- Look for the solution in the the form

$$f(t, X) = a_0(t) + a_1(t)X + \frac{a_2(t)}{2}X^2$$

- Substitute $f(t, X)$ into the PDE

$$a_0'(t) + a_1'(t)X + \frac{a_2'(t)}{2}X^2 - \kappa(X - \overline{X})(a_1(t) + a_2(t)X) + \frac{\sigma^2}{2}Xa_2(t) = 0$$

- Collect terms with different powers of X, zero, one and two, to get

$$a_0'(t) + \kappa\overline{X}a_1(t) = 0$$

$$a_1'(t) - \kappa a_1(t) + \left(\frac{\sigma^2}{2} + \kappa\overline{X}\right)a_2(t) = 0$$

$$a_2'(t) - 2\kappa a_2(t) = 0$$

with initial conditions

$$a_0(T) = a_1(T) = 0, \quad a_2(T) = 2$$

Example: Square-Root Diffusion

- We solve the system of equations starting from the third one and working up to the first:

$$a_0(t) = \overline{X}A\left(\frac{1}{2}e^{2\kappa(t-T)} - e^{\kappa(t-T)} + \frac{1}{2}\right)$$

$$a_1(t) = A\left(e^{\kappa(t-T)} - e^{2\kappa(t-T)}\right)$$

$$a_2(t) = 2e^{2\kappa(t-T)}$$

$$A = \frac{\sigma^2 + 2\kappa\overline{X}}{\kappa}$$

- Compare the exact expression above to an approximate expression, obtained by assuming that $T - t = \Delta t$ is small.

Example: Square-Root Diffusion

- Assume $T - t = \Delta t$ is small. Using Taylor expansion,

$$a_0(t) = o(\Delta t), \quad a_1(t) = A\kappa\Delta t + o(\Delta t)$$
$$a_2(t) = 2(1 - 2\kappa\Delta t) + o(\Delta t)$$

Then

$$\mathrm{E}_t(X_{t+\Delta t}^2 | X_t = X) \approx X^2 + \Delta t \left((\sigma^2 + 2\kappa\overline{X})X - 2\kappa X^2 \right)$$

- Alternatively,

$$X_{t+\Delta t} \approx X_t - \kappa(X_t - \overline{X})\,\Delta t + \sigma\sqrt{X_t}\sqrt{\Delta t}\varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, 1)$$

Therefore

$$\mathrm{E}_t(X_{t+\Delta t}^2 | X_t = X) \approx X^2 + \Delta t \left(\sigma^2 X - 2\kappa X(X - \overline{X}) \right)$$

Outline

Risk-Neutral Probability Measure

- Under the risk-neutral probability measure **Q**, expected conditional asset returns must equal the risk-free rate.
- Alternatively, using the discounted cash flow formula, the price P_t of an asset with payoff H_T at time T is given by

$$P_t = \mathrm{E}_t^{\mathbf{Q}} \left[\exp\left(-\int_t^T r_s \, ds \right) H_T \right]$$

  where r_s is the instantaneous risk-free interest rate at time s.
- The DCF formula under the risk-neutral probability can be used to compute asset prices by Monte Carlo simulation (e.g., using an Euler scheme to approximate the solutions of SDEs).
- Alternatively, one can derive a PDE characterizing asset prices using the connection between PDEs and SDEs.

Change of Measure

- We often want to consider a probability measure **Q** different from **P**, but the one that agrees with **P** on which events have zero probability (equivalent to **P**) (e.g., **Q** could be a risk-neutral measure).
- Different probability measures assign different relative likelihoods to the trajectories of the Brownian motion.
- It is easy to express a new probability measure **Q** using its density

$$\xi_T = \left(\frac{d\mathbf{Q}}{d\mathbf{P}}\right)_T$$

- For any random variable X_T,

$$\mathrm{E}_0^{\mathbf{Q}}[X_T] = \mathrm{E}_0^{\mathbf{P}}[\xi_T X_T], \quad \mathrm{E}_t^{\mathbf{Q}}[X_T] = \mathrm{E}_t^{\mathbf{P}}\left[\frac{\xi_T}{\xi_t} X_T\right], \ \xi_t \equiv \mathrm{E}_t^{\mathbf{P}}[\xi_T]$$

- **Q** is equivalent to **P** if ξ_T is positive (with probability one).

Change of Measure

- If we consider a probability measure **Q** different from **P**, but the one that agrees with **P** on which events have zero probability, then the **P**-Brownian motion $Z_t^{\mathbf{P}}$ becomes an Ito process under **Q**:

$$dZ_t^{\mathbf{P}} = dZ_t^{\mathbf{Q}} - \eta_t \, dt$$

  for some η_t. $Z_t^{\mathbf{Q}}$ is a Brownian motion under **Q**.
- When we change probability measures this way, only the drift of the Brownian motion changes, not the variance.
- Intuition: a probability measure assigns relative likelihood to different trajectories of the Brownian motion. Variance of the Ito process can be recovered from the shape of a single trajectory (quadratic variation), so it does not depend on the relative likelihood of the trajectories, hence, does not depend on the choice of the probability measure.

Risk-Neutral Probability Measure

- Under the risk-neutral probability measure, expected conditional asset returns must equal the risk-free rate.
- Start with the stock price process under **P**:

$$dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dZ_t^{\mathbf{P}}$$

- Under the risk-neutral measure **Q**,

$$dS_t = r_t S_t \, dt + \sigma_t S_t \, dZ_t^{\mathbf{Q}}$$

Thus, if $dZ_t^{\mathbf{P}} = -\eta_t \, dt + dZ_t^{\mathbf{Q}}$,

$$\mu_t - \sigma_t \eta_t = r_t$$

η_t is the price of risk.

- The risk-neutral measure **Q** is such that the process $Z_t^{\mathbf{Q}}$ defined by

$$dZ_t^{\mathbf{Q}} = \frac{\mu_t - r_t}{\sigma_t} \, dt + dZ_t^{\mathbf{P}}$$

is a Brownian motion under **Q**.

Risk-Neutral Probability Measure

- Under the risk-neutral probability measure, expected conditional asset returns must equal the risk-free rate.
- We conclude that for any asset (paying no dividends), the conditional risk premium is given by

$$\mathrm{E}_t^{\mathbf{P}}\left[\frac{dS_t}{S_t}\right] - r_t\,dt = \mathrm{E}_t^{\mathbf{P}}\left[\frac{dS_t}{S_t}\right] - \mathrm{E}_t^{\mathbf{Q}}\left[\frac{dS_t}{S_t}\right]$$

- Thus, mathematically, the risk premium is the difference between expected returns under the **P** and **Q** probabilities.

Risk-Neutral Probability Measure

- If we want to connect **Q** to **P** explicitly, how can we compute the density, $d\mathbf{Q}/d\mathbf{P}$?
- In discrete time, the density was conditionally lognormal.
- The density $\xi_T = (d\mathbf{Q}/d\mathbf{P})_T$ is given by

$$\xi_t = \exp\left(-\int_0^t \eta_u \, dZ_u^{\mathbf{P}} - \frac{1}{2}\int_0^t \eta_u^2 \, du\right), \quad 0 \leqslant t \leqslant T$$

- The state-price density is given by

$$\pi_t = \exp\left(\int_0^t -r_u \, du\right) \xi_t$$

- The reverse is true: if we define ξ_T as above, for any process η_t satisfying certain regularity conditions (e.g., η_t is bounded, or satisfies the Novikov's condition as in Back 2005, Appendix B.1), then measure **Q** is equivalent to **P** and

$$Z_t^{\mathbf{Q}} = Z_t^{\mathbf{P}} + \int_0^t \eta_u \, du$$

is a Brownian motion under **Q**.

Risk-Neutral Probability and Arbitrage

- If there exists a risk-neutral probability measure, then the model is arbitrage-free.
- If there exists a unique risk-neutral probability measure in a model, then all options are redundant and can be replicated by trading in the underlying assets and the risk-free bond.
- A convenient way to build arbitrage-free models is to describe them directly under the risk-neutral probability.
- One does not need to describe the **P** measure explicitly to specify an arbitrage-free model.
- However, to estimate models using historical data, particularly, to estimate risk premia, one must specify the price of risk, i.e., the link between **Q** and **P**.

Outline

Black-Scholes Model

- Assume that the stock pays no dividends and the stock price follows

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dZ_t^{\mathbf{P}}$$

- Assume that the interest rate is constant, r.
- Under the risk-neutral probability **Q**, the stock price process is

$$\frac{dS_t}{S_t} = r\, dt + \sigma\, dZ_t^{\mathbf{Q}}$$

Terminal stock price S_T is lognormally distributed:

$$\ln S_T = \ln S_0 + \left(r - \frac{\sigma^2}{2} \right) T + \sigma \sqrt{T} \varepsilon^{\mathbf{Q}}, \quad \varepsilon^{\mathbf{Q}} \sim \mathcal{N}(0,1)$$

- Price of any European option with payoff $H(S_T)$ can be computed as

$$P_t = \mathrm{E}_t^{\mathbf{Q}} \left[e^{-r(T-t)} H(S_T) \right]$$

Term Structure of Interest Rates

- Consider the Vasicek model of bond prices.
- A single-factor arbitrage-free model.
- To guarantee that the model is arbitrage-free, build it under the risk-neutral probability measure.
- Assume the short-term risk-free rate process under **Q**

$$dr_t = -\kappa(r_t - \bar{r})\,dt + \sigma\,dZ_t^{\mathbf{Q}}$$

- Price of a pure discount bond maturing at T is given by

$$P(t,T) = \mathrm{E}_t^{\mathbf{Q}}\left[\exp\left(-\int_t^T r_s\,ds\right)\right]$$

- Characterize $P(t,T)$ as a solution of a PDE.

Term Structure of Interest Rates

- Look for $P(t, T) = f(t, r_t)$.
- Using Ito's lemma,

$$\mathrm{E}_t^{\mathbf{Q}}[df(t, r_t)] = \left(\frac{\partial f(t, r_t)}{\partial t} - \kappa(r_t - \bar{r}) \frac{\partial f(t, r_t)}{\partial r_t} + \frac{1}{2}\sigma^2 \frac{\partial^2 f(t, r_t)}{\partial r_t^2} \right) dt$$

- Risk-neutral pricing requires that

$$\mathrm{E}_t^{\mathbf{Q}}[df(t, r_t)] = r_t f(t, r_t)\, dt$$

and therefore $f(t, r_t)$ must satisfy the PDE

$$\frac{\partial f(t, r)}{\partial t} - \kappa(r - \bar{r}) \frac{\partial f(t, r)}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 f(t, r)}{\partial r^2} = rf(t, r)$$

with the boundary condition

$$f(T, r) = 1$$

Term Structure of Interest Rates

- Look for the solution in the form

$$f(t, r_t) = \exp\left(-a(T-t) - b(T-t)r_t\right)$$

- Derive a system of ODEs on $a(t)$ and $b(t)$ to find

$$a(T-t) = \bar{r}(T-t) - \frac{\bar{r}}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) - \frac{\sigma^2}{4\kappa^3}\left(2\kappa(T-t) - e^{-2\kappa(T-t)} + 4e^{-\kappa(T-t)} - 3\right)$$

$$b(T-t) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)$$

Term Structure of Interest Rates

- Assume a constant price of risk η. What does this imply for the interest rate process under the physical measure **P** and for the bond risk premia?
- Use the relation

$$dZ_t^{\mathbf{P}} = -\eta\, dt + dZ_t^{\mathbf{Q}}$$

to derive

$$dr_t = -\kappa(r_t - \bar{r})\, dt + \sigma\eta\, dt + \sigma\, dZ_t^{\mathbf{P}} = -\kappa\left(r_t - \left(\bar{r} + \frac{\sigma\eta}{\kappa}\right)\right) dt + \sigma\, dZ_t^{\mathbf{P}}$$

- Expected bond returns satisfy

$$\mathrm{E}_t^{\mathbf{P}}\left(\frac{dP(t,T)}{P(t,T)}\right) = (r_t + \sigma_t^P \eta)\, dt$$

where

$$\sigma_t^P = \frac{1}{f(t,r_t)} \frac{\partial f(t,r_t)}{\partial r_t}\sigma = -b(T-t)\sigma$$

- $|b(T-t)\sigma|$ is the volatility of bond returns.

Equity Options with Stochastic Volatility

- Consider again the Heston's model. Assume that under the risk-neutral probability $\mathbf{Q}$, stock price is given by

$$d\ln S_t = (r - \frac{1}{2}v_t)\,dt + \sqrt{v_t}dZ_t^{\mathbf{Q},S}$$

$$dv_t = -\kappa(v_t - \overline{v})\,dt + \gamma\rho\sqrt{v_t}dZ_t^{\mathbf{Q},S} + \gamma\sqrt{1-\rho^2}\sqrt{v_t}dZ_t^{\mathbf{Q},v}$$

$$dZ_t^{\mathbf{Q},S}\,dZ_t^{\mathbf{Q},v} = 0$$

- $Z_t^{\mathbf{Q},v}$ models volatility shocks uncorrelated with stock returns.
- Constant interest rate r.
- The price of a European option with a payoff $H(S_T)$ can be computed as

$$P_t = \mathrm{E}_t^{\mathbf{Q}}\left[\exp\left(-r(T-t)\right)H(S_T)\right]$$

- We haven't said anything about the physical process for stochastic volatility. In particular, how is volatility risk priced?

Equity Options with Stochastic Volatility

- In the Heston's model, assume that the price of volatility risk is constant, η^v, and the price of stock price risk is constant, η^S.
- Then, under **P**, stock returns follow

$$d \ln S_t = \left(r + \eta^S \sqrt{v_t} - \frac{1}{2} v_t \right) dt + \sqrt{v_t} dZ_t^{\mathbf{P},S}$$

$$dv_t = \left(-\kappa (v_t - \overline{v}) + \gamma \sqrt{v_t} \left(\rho \eta^S + \sqrt{1-\rho^2} \eta^v \right) \right) dt + \gamma \rho \sqrt{v_t} dZ_t^{\mathbf{P},S} + \gamma \sqrt{1-\rho^2} \sqrt{v_t} dZ_t^{\mathbf{P},v}$$

$$dZ_t^{\mathbf{P},S} \, dZ_t^{\mathbf{P},v} = 0$$

We have used

$$dZ_t^{\mathbf{P},S} = -\eta^S \, dt + dZ_t^{\mathbf{Q},S}$$

$$dZ_t^{\mathbf{P},v} = -\eta^v \, dt + dZ_t^{\mathbf{Q},v}$$

- Conditional expected excess stock return is

$$\mathrm{E}_t^{\mathbf{P}} \left[\frac{dS_t}{S_t} - r \, dt \right] = \mathrm{E}_t^{\mathbf{P}} \left[d \ln S_t + \frac{v_t}{2} dt - r \, dt \right] = \left(\eta^S \sqrt{v_t} \right) \, dt$$

Equity Options with Stochastic Volatility

- Our assumptions regarding the market prices of risk translate directly into implications for return predictability.
- For stock returns, our assumption of constant price of risk predicts a nonlinear pattern in excess returns: expected excess stock returns proportional to conditional volatility.
- Suppose we construct a position in options with the exposure λ_t to stochastic volatility shocks and no exposure to the stock price:

$$dW_t = [...] \, dt + \lambda_t dZ_t^{\mathbf{P},v}$$

Then the conditional expected gain on such a position is

$$\left(W_t r + \lambda_t \eta^v\right) dt$$

Heston's Model of Stochastic Volatility

- Assume that under the risk-neutral probability $\mathbf{Q}$, stock price is given by

$$
\begin{aligned}
d \ln S_t &= (r - \frac{1}{2} v_t)\, dt + \sqrt{v_t} dZ_t^{\mathbf{Q},S} \\
dv_t &= -\kappa(v_t - \overline{v})\, dt + \gamma\rho\sqrt{v_t} dZ_t^{\mathbf{Q},S} + \gamma\sqrt{1-\rho^2}\sqrt{v_t} dZ_t^{\mathbf{Q},v} \\
dZ_t^{\mathbf{Q},S}\, dZ_t^{\mathbf{Q},v} &= 0
\end{aligned}
$$

- $Z_t^{\mathbf{Q},v}$ models volatility shocks uncorrelated with stock returns.
- Constant interest rate r.
- The price of a European option with a payoff $H(S_T)$ can be computed as

$$
P_t = \mathrm{E}_t^{\mathbf{Q}} \left[\exp\left(-r(T-t)\right) H(S_T) \right]
$$

- Assume that the price of volatility risk is constant, η^v, and the price of stock price risk is constant, η^S.

Variance Swap in Heston's Model

- Consider a variance swap, paying

$$\int_t^T (d \ln S_u)^2 - K_t^2$$

at time T. What should be the strike price of the swap, K_t, to make sure that the market value of the swap at time t is zero?

- Using the result on quadratic variation,

$$(d \ln S_t)^2 = v_t \, dt$$

the strike price must be such that

$$e^{-r(T-t)} \mathrm{E}_t^{\mathbf{Q}} \left[\left(\int_t^T v_u \, du - K_t^2 \right) \right] = 0$$

- Need to compute

$$K_t^2 = \mathrm{E}_t^{\mathbf{Q}} \left[\int_t^T v_u \, du \right]$$

Variance Swap in Heston's Model

- Since

$$v_u = v_t - \int_t^u \kappa(v_s - \overline{v})\, ds + \gamma\rho \int_t^u \sqrt{v_s}\, dZ_s^{\mathbf{Q},S} + \gamma\sqrt{1-\rho^2} \int_t^u \sqrt{v_s}\, dZ_s^{\mathbf{Q},v}$$

  we find that

$$\mathrm{E}_t^{\mathbf{Q}}[v_u] = v_t - \mathrm{E}_t^{\mathbf{Q}}\left[\int_t^u \kappa(v_s - \overline{v})\, ds\right] = v_t - \int_t^u \kappa(\mathrm{E}_t^{\mathbf{Q}}[v_s] - \overline{v})\, ds$$

- Solving the above equation for $\mathrm{E}_t^{\mathbf{Q}}[v_u]$, we find

$$\mathrm{E}_t^{\mathbf{Q}}[v_u] = \overline{v} + e^{-\kappa(u-t)}(v_t - \overline{v})$$

- We obtain the strike price

$$K_t^2 = \mathrm{E}_t^{\mathbf{Q}}\left[\int_t^T v_u\, du\right] = \overline{v}(T-t) + (v_t - \overline{v})\frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)$$

Expected Profit/Loss on a Variance Swap

- To compute expected profit/loss on a variance swap, we need to evaluate

$$\mathrm{E}_t^{\mathbf{P}}\left[\int_t^T (d\ln S_u)^2 - K_t^2\right]$$

- Instantaneous expected gain on a long position in the swap is easy to compute in closed form.
- The market value of a swap starts at 0 at time t, and at $s > t$ becomes

$$P_s \equiv \mathrm{E}_s^{\mathbf{Q}}\left[e^{-r(T-s)}\left(\int_t^s v_u\,du + \int_s^T v_u\,du - K_t^2\right)\right]$$
$$= e^{-r(T-s)}\left(\int_t^s v_u\,du + K_s^2 - K_t^2\right)$$

- We conclude that the instantaneous gain on the long swap position at time s equals

$$[\ldots]\,ds + \frac{1}{\kappa}e^{-r(T-s)}\left(1 - e^{-\kappa(T-s)}\right)dv_s$$

Expected Profit/Loss on a Variance Swap

- We conclude that the instantaneous gain on the long swap position at time s equals

$$[...]\, ds + \frac{1}{\kappa} e^{-r(T-s)} \left(1 - e^{-\kappa(T-s)}\right) dv_s$$

- Given our assumed market prices of risk, η^S and η^v, the time-s expected instantaneous gain on the swap opened at time t is

$$P_s r\, ds + \gamma \sqrt{v_s} \frac{1}{\kappa} e^{-r(T-s)} \left(1 - e^{-\kappa(T-s)}\right) \left(\rho \eta^S + \sqrt{1-\rho^2} \eta^v\right) ds$$

Summary

- Risk-neutral pricing is a convenient framework for developing arbitrage-free pricing models.
- Connection to classical results: risk-neutral expectation can be characterized by a PDE.
- Risk premium on an asset is the difference between expected return under **P** and under **Q** probability measures.

Readings

- Back 2005, Sections 2.1-2.6, 2.8-2.9, 2.11, 13.2, 13.3, Appendix B.1.