M.I.T. **15.450-Fall 2010**
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Examples of Dynamic Programming Problems

Problem 1 *A given quantity X of a single resource is to be allocated optimally among N production processes. Each process produces an output of the same good in the amount $\sqrt{x}$, where x is the amount of input (x has to be nonnegative). Use dynamic programming to determine the allocation of the resource x_n^*, $n = 1, \ldots, N$, among the production processes that maximizes the aggregate output.*

1. *Enumerate the production processes 1 to n. Suppose that certain amount of the resource has been already allocated among the first $n-1$ processes. Let the remaining amount be X_n. Let $J_n(X_n)$ be the aggregate output of the remaining $N-n$ production processes, given that the input X_n is distributed optimally among them. Show by induction that the value function $J_n(X_n)$ has the form $J_n(X_n) = c_n\sqrt{X_n}$. Calculate the constants c_n for $n = 1, \ldots, N$.*

2. *Use the fact that $X_1 = X$ to find x_n^*, $n = 1, \ldots, N$, and the optimal aggregate output J_1^*.*

Problem 2 *Consider the following dynamic investment problem. The market consists of two assets: the riskless asset and the risky asset. Both assets are traded periodically at time periods $t = 0, 1, \ldots, T$. The net simple return over a single holding period on the riskless asset is denoted by R_{ft} (e.g., $R_{ft} = 5\%$), while the net simple return on the risky asset is denoted by R_t. It is assumed that the distribution of the returns on the risky asset is given by*

$$R_t = \mu + \sigma\epsilon_t,$$

where ϵ_t, $t = 0, 1, \ldots, T-1$ are independently and identically distributed standard normal random variables, i.e., $\epsilon_t \sim \mathcal{N}(0, 1)$.

Investor seeks to maximize the expected utility of wealth at time T. Her utility function is exponential: $U(x) = -e^{-\gamma x}$. The initial wealth is denoted by W_0. There are no constraints on short-sales and borrowing and there are no transactions costs.

1. *Let the control variable be x_t – the amount of wealth invested in the risky asset at time t. Express the wealth W_{t+1} at time $t+1$ as a function of the wealth W_t at time t and x_t.*

2. *Let $J_t(W_t)$ denote the value function (the indirect utility function) of wealth W_t at time t. Show using induction that the value function has the functional form*

$$J_t(x) = -a_t e^{-b_t x}.$$

Show that the optimal amount of wealth allocated into the risky asset, x_t^*, does not depend on the current level of wealth W_t.

3. *Find a recursive relation for a_t and b_t. What is the optimal investment strategy x_t^*, $t = 0, 1, \dots, T-1$? How does it depend on the risk-aversion parameter γ, the mean and the variance of the returns on the risky asset and on the investment horizon $T - t$?*

Solution of Problem 1

1. When $n = N$, all the remaining resource X_N should be allocated into a single remaining production process N, i.e.,
$$J_N(X_N) = \sqrt{X_N}.$$
Thus, $c_N = 1$.
Let's assume that for $n \geq k+1$, $J_n(X_n) = c_n\sqrt{X_n}$. Note that $X_{k+1} = X_k - x_k$. Therefore, according to the Bellman optimality principle,
$$J_k(X_k) = \max_{x_k} \left(\sqrt{x_k} + J_{k+1}(X_k - x_k)\right), \ s.t. \ x_k \leq X_k. \quad (1)$$
The solution of this problem x_k^* can be found from the first-order condition
$$\frac{1}{2\sqrt{x_k^*}} + \frac{c_{k+1}}{2\sqrt{X_k - x_k^*}} = 0. \quad (2)$$
From (2) we find that
$$x_k^*(X_k) = \frac{X_k}{1 + c_{k+1}^2}. \quad (3)$$
Then, according to (1),
$$J_k(X_k) = \sqrt{1 + c_{k+1}^2}\sqrt{X_k}.$$
Thus, $c_k = \sqrt{1 + c_{k+1}^2}$ and we conclude the induction argument.
In order to find all constants c_n explicitly, note that $c_N = 1$ and $c_n^2 = 1 + c_{n+1}^2$. Therefore, $c_n^2 = N - n + 1$ and $c_n = \sqrt{N - n + 1}$.

2. Using the relation $X_{n+1}^* = X_n^* - x_n^*(X_n^*)$ and 3, we conclude that
$$X_{n+1}^* = \frac{c_{n+1}^2}{1 + c_{n+1}^2} X_n^* = \frac{N-n}{N-n+1} X_n^*.$$
Since $X_1^* = X$,
$$X_n^* = \frac{N-1}{N}\frac{N-2}{N-1}\cdots\frac{N-n+1}{N-n+2}X = \frac{N-n+1}{N}X.$$

We use (3) again to conclude that

$$x_n^* = \frac{1}{1+N-k}\frac{N-k+1}{N}X = \frac{X}{N}.$$

Thus, the resource has to be allocated evenly among all N production processes. Also $J_1^* = c_1\sqrt{X_1^*} = \sqrt{N}\sqrt{X}$.

Solution of Problem 2

Let Z_t denote the excess return on the risky asset, i.e.,

$$Z_t = R_t - R_{ft}.$$

1. If x_t is the amount of wealth allocated into the risky asset, $W_t - x_t$ must be allocated into the riskless asset. Then

$$W_{t+1} = x_t(1+R_t) + (W_t - x_t)(1+R_{ft}) = W_t(1+R_{ft}) + x_t Z_t. \tag{4}$$

2. When $t = T$, $J_T = -e^{-\gamma W_T}$. Thus, $a_T = 1$ and $b_t = \gamma$.

   Let's assume that for $n \geq t+1$, $J_n(W_n) = -a_n e^{-b_n W_n}$. Then, according to the Bellman optimality principle and (4),

$$J_t(W_t) = \max_{x_t} \mathbf{E}\left[-a_t \exp(-b_t W_t(1+R_{ft}) - b_t x_t Z_t)\right].$$

   Next, note that

$$\mathbf{E}\left[\exp(-b_t x_t Z_t)\right] = \mathbf{E}\left[\exp(-b_t x_t(\mu - R_{ft} + \sigma\epsilon_t))\right] = \exp\left(-b_t x_t(\mu - R_{ft}) + \frac{1}{2}\sigma^2 b_t^2 x_t^2\right).$$

   Thus,

$$J_t(W_t) = \max_{x_t} \mathbf{E}\left[-a_t \exp\left(-b_t W_t(1+R_{ft}) - b_t x_t(\mu - R_{ft}) + \frac{1}{2}\sigma^2 b_t^2 x_t^2\right)\right].$$

   The first-order condition for the maximization problem is equivalent to

$$-b_t(\mu - R_{ft}) + \sigma^2 b_t^2 x_t^* = 0,$$

   which implies that

$$x_t^*(W_t) = \frac{\mu - R_{ft}}{\sigma^2 b_t}. \tag{5}$$

   Thus, the optimal amount of wealth allocated into the risky asset x_t^* does not depend on the current level of wealth W_t.

   We find that

$$J_t(W_t^*) = -a_t \exp\left(-b_t W_t^*(1+R_{ft}) - \frac{1}{2}\frac{(\mu - R_{ft})^2}{\sigma^2}\right), \tag{6}$$

   which concludes the step of induction.

3. From (6) we observe that

$$a_t = a_{t+1} \exp\left(-\frac{1}{2}\frac{(\mu - R_{ft})^2}{\sigma^2}\right), \quad b_t = b_{t+1}(1 + R_{ft}).$$

We can solve these recursive equations using the terminal conditions

$$a_T = 1, \quad b_T = \gamma$$

to obtain

$$a_t = \gamma \exp\left(-\frac{T-t}{2}\frac{(\mu - R_{ft})^2}{\sigma^2}\right), \quad b_t = (1 + R_{ft})^{T-t}.$$

We now combine this with (5) to obtain

$$x_t^* = \frac{1}{\gamma}\frac{\mu - R_{ft}}{\sigma^2}(1 + R_{ft})^{-(T-t)}.$$

From this we conclude that x_t^* is higher when the risk-aversion parameter γ is lower; it increases linearly with the mean excess return on the riskless asset and it is inversely proportional to the variance of the returns; it is lower for longer investment horizons $T - t$.