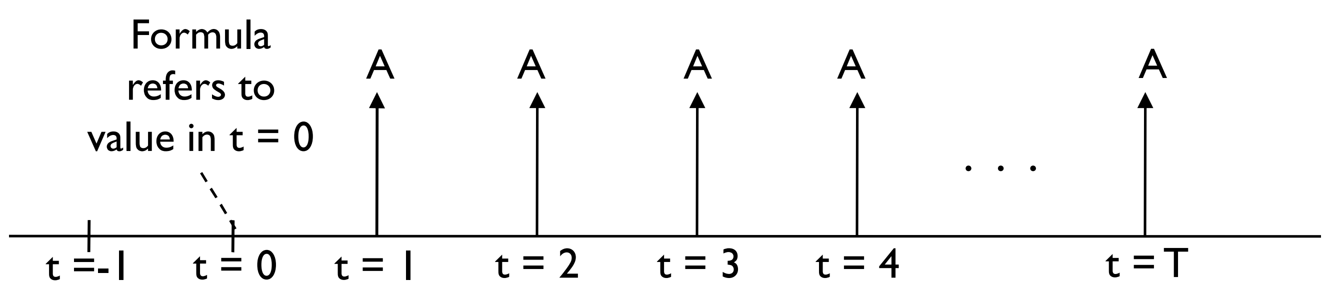
15.401 Recitation

Extra Session: Mid-Term Review

General Advice

- Show your work! Answers only give you partial credit
- Write down the formulas you use
- Draw timelines for cash flows
- Make sure you apply the annuity/ perpetuity formulas correctly – Example: $\text{PV (Annuity)} = A \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^T}\right]$



- State your assumptions
- Leave plenty of decimal places for interest rates (e.g., 1.2345%)

Sample Midterm Solutions

- Q – Compounding
- Q – APR-EAR Conversion
- Q – Common

Q1 – Compounding

- If the annual interest rate is 10 percent, how long would you have to wait before a $17,500 investment doubles in value?

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- If the annual interest rate is 10 percent, how long would you have to wait before a $17,500 investment doubles in value?

Advice:

- Question whether the rate is expressed as EAR, APR, or other; and what is the compounding period
- If some of the above is not clear to you, state your assumptions
- Write down the equation in terms of *T*
- Compute precise value, then round up
- Show your work!

Q1 – Compounding

- Let *T* be the amount of time (in years) required, then

$$17{,}500 \cdot (1+10\%)^T = 17{,}500 \cdot 2$$

$$1.1^T = 2$$

$$T \cdot Ln(1.1) = Ln(2)$$

$$T = \frac{\ln 2}{\ln 1.1} = 7.2725$$

- The minimum number of years is 8.

Q2 – APR-EAR Conversion

- Your car dealer offers you a loan for part of the purchase price of a new car, citing an annual percentage rate (APR) of 8.5%. What is the effective annual rate of such a loan (recall that an auto loan typically requires monthly payments)?

Q2 – APR-EAR Conversion

- Your car dealer offers you a loan for part of the purchase price of a new car, citing an annual percentage rate (APR) of 8.5%. What is the effective annual rate of such a loan (recall that an auto loan typically requires monthly payments)?

<u>Advice</u>:

- Know how to EAR⇔APR

$$\text{Effective Rate per Compounding Interval} = \frac{r_{APR}}{k}$$

$$r_{\text{EAR}} = \left(1 + \frac{r_{\text{APR}}}{k}\right)^k - 1.$$

APR: annual percentage rate
EPR: effective annual rate
k: # comp interval per period

- Leave at least 2 decimal places in the end

Q2 – APR-EAR Conversion

- Given monthly-compounded APR, we have

$$r_{EAR} = \left(1 + \frac{r_{APR}}{k}\right)^k - 1$$

$$= \left(1 + \frac{8.5\%}{12}\right)^{12} - 1$$

$$= 8.8391\%$$

Q3 – Common Stocks

- Company ABC has just paid a dividend of 50 cents per share. Because of its growth potential, its dividend is forecasted to grow at a rate of 7 percent per year indefinitely. If the company's appropriate cost of capital (given its risk) is 11 percent, what was ABC's share price immediately *before* it paid its 50 cent dividend, i.e the stock price right before the ex-dividend date?

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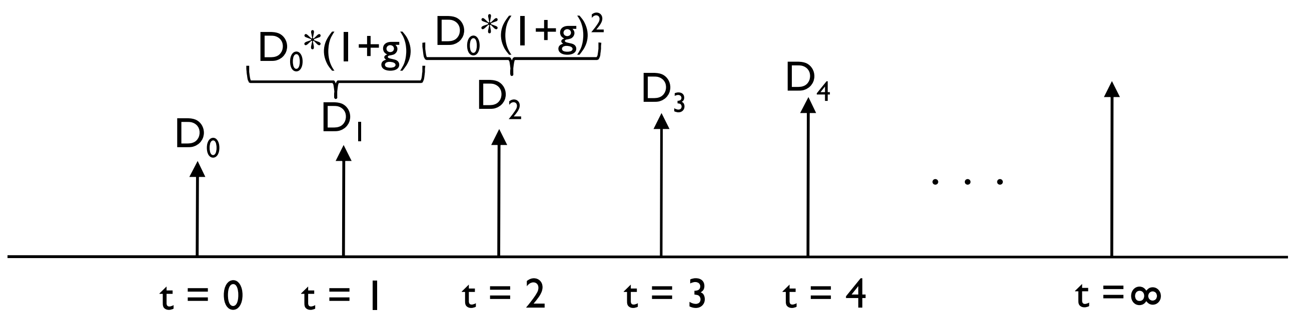
Advice:

- Be careful with the times when dividend payment occurs – draw a timeline!
- In this case, remember to include D_0
- Write the growing perpetuity formula
- Use the formula correctly – numerator is D_1, not D_0

Q3 – Common Stocks

- Parameters:

$$D_0 = 0.50; D_1 = 0.50 \times 1.07; g = 0.07; r = 0.11$$



- Dividend discount model:

$$P_0 = D_0 + \frac{D_1}{r-g}$$

$$= 0.50 + \frac{0.50 \times 1.07}{0.11 - 0.07}$$

$$= \$13.88$$

Q3 – Common Stocks

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Q4 – Fixed Income Securities

- The current market price of a two-year 25 percent coupon bond with a $1,000 face value is $1,219.71 (recall that such a bond pays coupons of $250 at the end of years 1 and 2, and the principal of $1,000 at the end of year 2). The current market price of a one-year pure discount bond with a $50 face value is $44.64.

  a) What must the price of a two-year pure discount bond with a $2,500 face value be in order to avoid arbitrage?

Advice:

- Do not confuse r_1 with r_2 and YTM
- In this case, do not discount the coupon with the YTM

Q4 – Fixed Income Securities

- "The current market price of a one-year pure discount bond with a \$50 face value is \$44.64":

$$\frac{50}{(1+r_1)} = 44.64 \Rightarrow r_1 = 12.0072\%$$

- "The current market price of a two-year 25 percent coupon bond with a \$1,000 face value is \$1,219.71":

$$\frac{250}{(1+r_1)} + \frac{1{,}250}{(1+r_2)^2} = 1{,}279.71$$

$$\frac{250}{(1+0.120072)} + \frac{1{,}250}{(1+r_2)^2} = 1{,}279.71 \Rightarrow r_2 = 11.9990\%$$

- Now we know the spot rates, r_1 and r_2

Q4 – Fixed Income Securities

- The price of the two-year pure discount bond must be

$$P = \frac{2{,}500}{(1+r_2)^2} = \frac{2{,}500}{(1+0.119990)^2} = \$1{,}993.02$$

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b)Suppose the price of the two-year pure discount bond with a $2,500 face value is only $1,900. Is there an arbitrage opportunity? Is yes, how would you structure a trade that has zero cash flow in years 1 and 2 and a positive cash flow only in year 0 (i.e. now).

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- There must be an arbitrage opportunity because $1,900 is not the fair price of $1,993.02 ! How to capitalize it?

Q4 – Fixed Income Securities

Advice:

- Consider all the instruments you may use – 3 in this example
- Build a table that shows for every year that are expected to come from each bond (see below)
- In this case, start by writing the cash flows for the bond that is not fairly priced, or the one with longer maturity
- Use your logic!

Bond	Position	CF at 0	CF at 1	CF at 2
1-yr, Zero Coupon, $50 Par, sells for $44.64				
2-year, 25% coupon, $1,000 Par, sells for $1,219.71				
2-yr, Zero Coupon, $2,500 Par, sells for $1,900				
	TOTAL:	+ ?	0	0

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2-year, 25% coupon, $1,000 Par, sells for $1,219.71				
2-yr, Zero Coupon, $2,500 Par, sells for $1,900	Buy (Long) 1 bond	-1,900		2,500
	TOTAL:	+ ?	0	0

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- Use your logic!

Bond	Position	CF at 0	CF at 1	CF at 2
1-yr, Zero Coupon, $50 Par, sells for $44.64				
2-year, 25% coupon, $1,000 Par, sells for $1,219.71	Sell (Short) 2 bonds	+2,439.42	-500	-2,500
2-yr, Zero Coupon, $2,500 Par, sells for $1,900	Buy (Long) 1 bond	-1,900		2,500
	TOTAL:	+ ?	0	0

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- Use your logic!

Bond	Position	CF at 0	CF at 1	CF at 2
1-yr, Zero Coupon, $50 Par, sells for $44.64	Buy (Long) 10 bonds	-446.40	+500	
2-year, 25% coupon, $1,000 Par, sells for $1,219.71	Sell (Short) 2 bonds	+2,439.42	-500	-2,500
2-yr, Zero Coupon, $2,500 Par, sells for $1,900	Buy (Long) 1 bond	-1,900		2,500
	TOTAL:	+ 93.02	0	0

Q4 – Fixed Income Securities

Bond	Position	CF at 0	CF at 1	CF at 2
1-yr, Zero Coupon, $50 Par, sells for $44.64	Buy (Long) 10 bonds	-446.40	+500	
2-year, 25% coupon, $1,000 Par, sells for $1,219.71	Sell (Short) 2 bonds	+2,439.42	-500	-2,500
2-yr, Zero Coupon, $2,500 Par, sells for $1,900	Buy (Long) 1 bond	-1,900		2,500
	TOTAL:	+ 93.02	0	0

Note:

- We can make free money today, with no risk – arbitrage!
- In our example the profit is $93.02, the same amount by which each 2-year, zero coupon, is underpriced
- There is more than solution to this problem, but all of them are multiples of this simple case

Q4 – Fixed Income Securities

Alternative method:

- Solve and equation system:

	CF at 0	CF at 1	CF at 2
n_1 x B_1	$-n_1$ x 1219.71	n_1 x 250	n_1 x 1250
n_2 x B_2	$-n_2$ x 44.64	n_2 x 50	0
n_3 x B_3	$-n_3$ x 1900	0	n_3 x 2500
	+	0	0

Q4 – Fixed Income Securities

- This portfolio is an arbitrage portfolio if

$$\begin{cases} w_1 \times 1250 + w_3 \times 2500 = 0 \\ w_1 \times 250 + w_2 \times 50 = 0 \\ -w_1 \times 1219.71 - w_2 \times 44.64 - w_3 \times 1900 > 0 \end{cases}$$

- Solution of the system:

$$\begin{cases} w_1 = -2k \\ w_2 = 10k \quad \text{for any } k > 0 \\ w_3 = k \end{cases}$$

Q5 – Present Value

- Your friend is celebrating her 35th birthday today and wants to start saving for her anticipated retirement at age 65 (she will retire on her 65th birthday). She would like to be able to withdraw $80,000 from her savings account on each birthday for at least 20 years following her retirement (the first withdrawal will be on her 66th birthday). Your friend intends to invest her money in the local savings bank which offers 4% per year. She wants to make equal annual deposits on each birthday in a new savings account she will establish for her retirement fund.

  If she starts making these deposits on her 36th birthday and continues to make deposits until she is 65 (the last deposit will be on her 65th birthday), what amount must she deposit annually to be able to make the desired withdrawals upon retirement?

Q5 – Present Value

Advice:

- Draw a timeline!
- Shift the timeline so that now is t = 0
- Write down the annuity formula
- Find out the correct discount rate for the cash flows, in this case 4%

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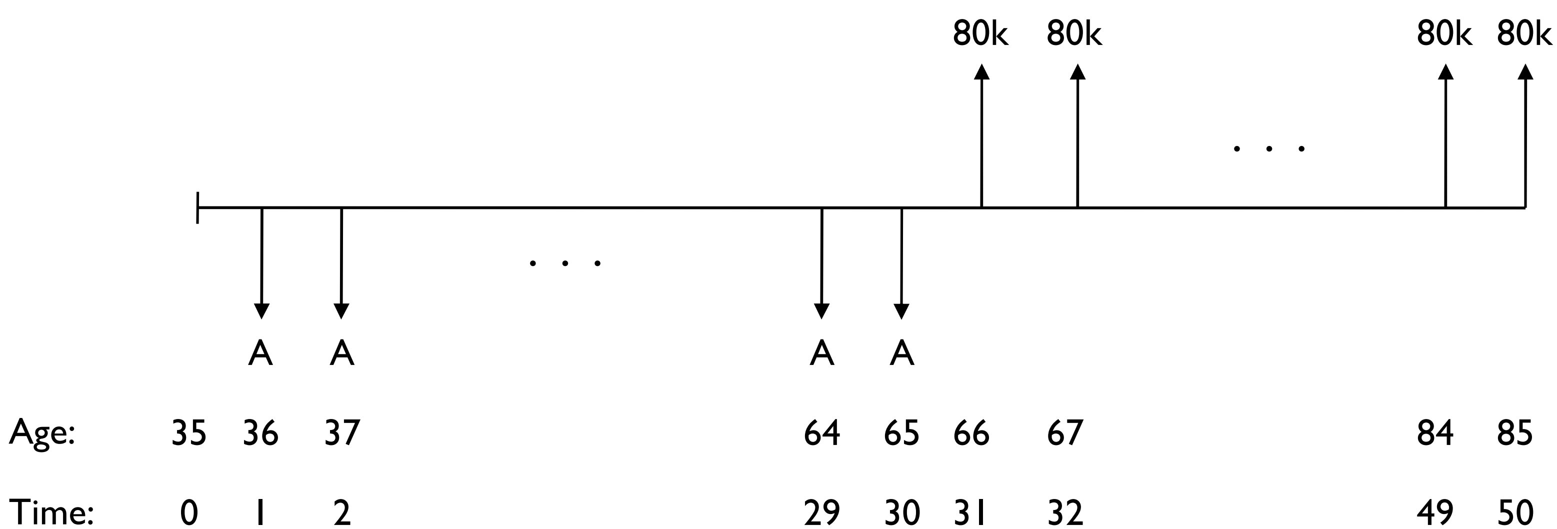

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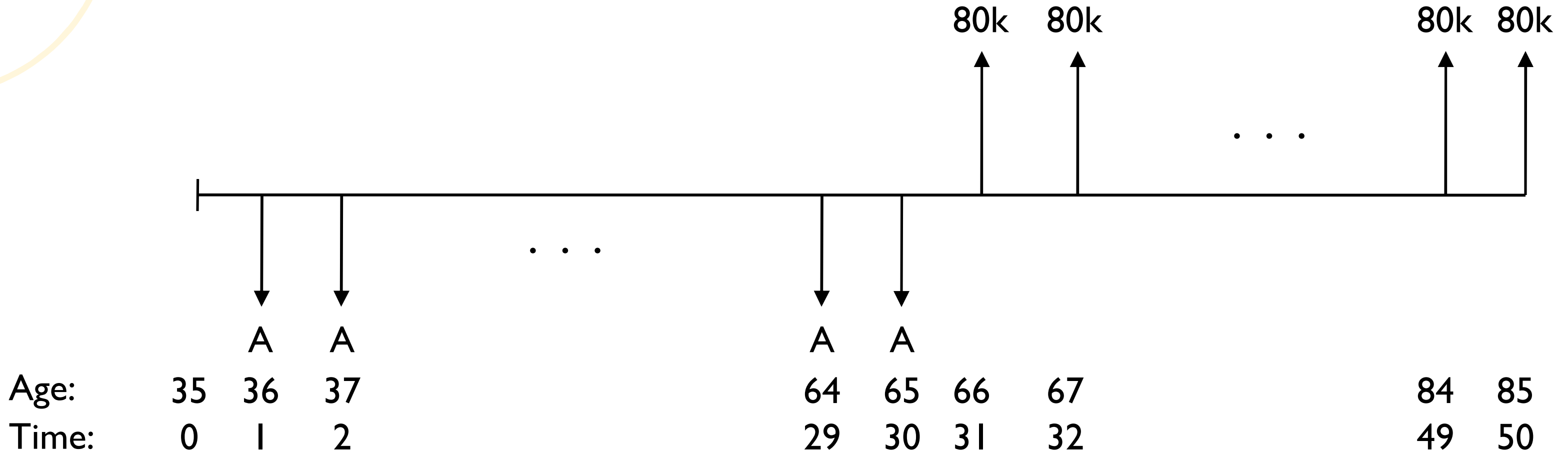


- Let "*A*" be the necessary annual deposit, then, using the annuity formulas:

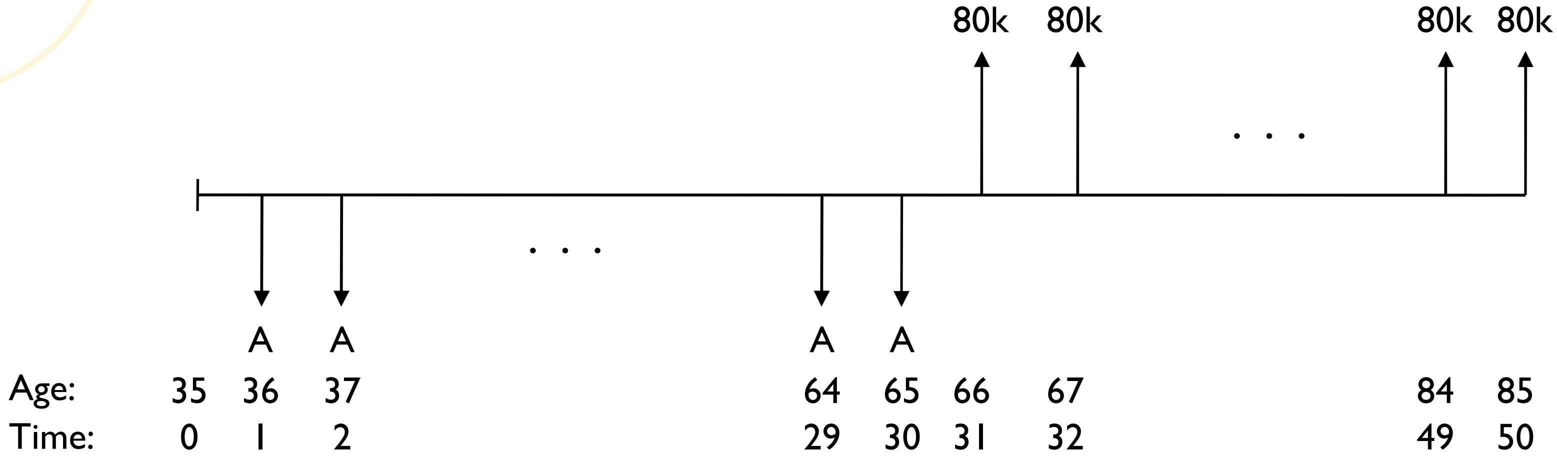
$$PV(withdraws) = PV(savings)$$

$$\frac{1}{(1.04)^{30}} \cdot \left[\frac{80{,}000}{0.04} \left(1 - \frac{1}{1.04^{20}} \right) \right] = \frac{A}{0.04} \left(1 - \frac{1}{1.04^{30}} \right)$$

$$335{,}212.11 = A \cdot 17.2920$$

$$A = \$19{,}385.35$$

Q5 – Present Value



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$$335{,}212{,}11 = A \cdot 19.3854$$

$$A = \$19{,}385.35$$

- Note that the $80K annuity has been discounted 30 years

Q6 – Forwards & Futures

- The current level of the S&P 500 is $1040. The risk-free interest rate per year is 2%. Assume negligible dividends. The 6 month futures contract is trading at $1060.

  a) Is there an arbitrage opportunity? Briefly explain.

<u>Advice</u>:

- Write down the spot-futures parity
- Make sure that "T", "r_f" and "y" are based on the same unit of time

Q6 – Forwards & Futures

- Spot-futures parity requires

$$F_T = S_0(1+r-y)^T$$

$$1040(1+2\%)^{0.5} = 1050.35 \neq 1060$$

- There is an arbitrage opportunity because the trading price is not equal to the "fair price" (calculated with the spot-futures parity)

b) If there is an arbitrage opportunity, what strategy would you use to exploit it without using any funds of your own?

Q6 – Forwards & Futures

Advice:

- Draw the CF table – similar to that in question 4, with all the instruments you have
- Note that you can always borrow or lend money at the risk free rate – eg. $1,040*(1+0.02)^{0.5}$ = $1,050.35
- Use your logic – And if you arrive to the inverse conclusion just inverse all the signs in the table

Security	Position	CF at 0	CF at 6 months
S&P 500 stock trading at $1,040			
Borrow / Lend Money			
6 month future trading at $1,060			
TOTAL:			

Q6 – Forwards & Futures

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Security	Position	CF at 0	CF at 6 months
S&P 500 stock trading at \$1,040			
Borrow / Lend Money			
6 month future trading at \$1,060	Short 1	0	1,060 – S_T
	TOTAL:	\$0	1,060 – S_T

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Security	Position	CF at 0	CF at 6 months
S&P 500 stock trading at $1,040	Buy 1	-1,040	S_T
Borrow / Lend Money			
6 month future trading at $1,060	Short 1	0	$1{,}060 - S_T$
	TOTAL:	-$1,040	$1,060

Q6 – Forwards & Futures

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Security	Position	CF at 0	CF at 6 months
S&P 500 stock trading at $1,040	Buy 1	-1,040	S_T
Borrow / Lend Money	Borrow $1,040	1,040	-1,050.35
6 month future trading at $1,060	Short 1	0	$1{,}060 - S_T$
	TOTAL:	$0	$9.65

Q6 – Forwards & Futures

Note:

- Arbitrage! In 6 months, we will have made $9.65 without any risk or investment

Security	Position	CF at 0	CF at 6 months
S&P 500 stock trading at $1,040	Buy 1	-1,040	S_T
Borrow / Lend Money	Borrow $1,040	1,040	-1,050.35
6 month future trading at $1,060	Short 1	0	$1,060 – S_T$
	TOTAL:	$0	$9.65

15.401 Finance Theory I
Fall 2008