15.093 - Recitation 1

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1 Convex Functions and Convex Sets

A set A is convex if $\forall x, y \in A$, we have

$$\lambda x + (1 - \lambda) y \in A.$$

A function $f : \mathbb{R} \to \mathbb{R}$ is convex if $\forall x, y \in \mathbb{R}$, we have

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y), \ \forall \lambda \in [0, 1].$$

2 Optimization Problems

$$\begin{array}{ll} \min & f(x) \\ \text{s/t} & x \in \mathcal{X}. \end{array}$$

Commonly $\mathcal{X} = \{x : g_i(x) \leq 0, \ i = 1, \ldots, m\}$.

2.1 Convex Optimization Problem

A convex optimization problem has:

- convex objective function f
- $\mathcal{X}$ a convex set, e.g. constraints $g_i(x) \leq 0$, where g_i is convex

2.2 Linear Optimization Problem

An LP (Linear Program) is a special case of a convex programming problem, and has:

- linear objective function $c'x$
- linear constraints $a_i'x \leq b_i$

  then $\mathcal{X}$ is a *polyhedron* – the intersection of a finite number of *halfspaces*

2.2.1 Standard Form

Any LP can be transformed into standard form:

$$\begin{array}{rrcl} \min & c'x & & \\ \text{s/t} & Ax & = & b \\ & x & \geq & 0 \end{array}$$

3 Some LP Modeling Tricks

3.1 Absolute Value Function

Minimizing $\sum_{j=1}^{n} c_j|x_j|$, where $c_j > 0,\ \forall j \in \{1, \ldots, n\}$, subject to $Ax = b$ can be written as the following LP:

$$\begin{array}{rrcl} \min & c'(x^+ + x^-) & & \\ \text{s/t} & A(x^+ - x^-) & = & b \\ & x^+, x^- & \geq & 0. \end{array}$$

Check for correctness: when $x_j^{+*} > 0,\ x_j^{-*} = 0$ and vice-versa. Why?

Alternatively, we can write

$$\begin{array}{rrcl} \min & c'z & & \\ \text{s/t} & z_j & \geq & x_j \\ & z_j & \geq & -x_j \\ & Ax & = & b. \end{array}$$

3.2 Piecewise Linear Convex Functions

Minimize $f(x)$ subject to $Ax = b,\ x \geq 0$, where

$$f = \begin{cases} c_1x - d_1, & x \leq u_1, \\ c_2x - d_2, & u_1 \leq x \leq u_2, \\ c_3x - d_3, & u_2 \leq x. \end{cases}$$

Letting for example $f(x) = \begin{cases} \frac{1}{2}x, & x \leq 2, \\ x - 1, & 2 \leq x \leq 3, \\ 3x - 7, & 3 \leq x, \end{cases}$ we can write

$$\begin{array}{rrcl} \min & z & & \\ \text{s/t} & z & \geq & \frac{1}{2}x \\ & z & \geq & x - 1 \\ & z & \geq & 3x - 7 \\ & Ax & = & b \\ & x & \geq & 0. \end{array}$$

Another interesting approach if x is scalar is to let $x = x^1 + x^2 + x^3$, and write

$$\begin{array}{lll} \min & \frac{1}{2}x^1 + x^2 + 3x^3 & \\ \text{s/t} & A(x^1 + x^2 + x^3) & = \quad b \\ & 0 \le x^1 \le 2,\ 0 \le x^2 \le 1,\ x^3 \ge 0. & \end{array}$$

Check for correctness: when $x^{i*} > 0,\ x^{i-1*} = u_{i-1}$. Why?

4 AMPL

AMPL (A Mathematical Programming Language) is a high-level programming language for writing and solving mathematical programs (linear and non-linear, in continuous and discrete variables). AMPL itself does not solve the problems, instead it calls an external solver (such as CPLEX) to solve the optimization problem.
An AMPL program consists of two parts:

- a model file (.mod file), and
- a data file (.dat file)

The model file writes the linear program using the grammar of AMPL and it defines various sets, parameters, variables, objective and constraints. The data file provides data for the model file.