15.093: Optimization Methods

Lecture 12: Discrete Optimization

1 Todays Lecture

Slide 1

- Modeling with integer variables
- What is a good formulation?
- Theme: The Power of Formulations

2 Integer Optimization

2.1 Mixed IO

Slide 2

$$\begin{array}{lll} \text{(MIO)} & \max & \boldsymbol{c}'\boldsymbol{x} + \boldsymbol{h}'\boldsymbol{y} \\ & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{b} \\ & & \boldsymbol{x} \in Z_+^n (\boldsymbol{x} \geq 0, \boldsymbol{x} \text{ integer}) \\ & & \boldsymbol{y} \in R_+^m (\boldsymbol{y} \geq 0) \end{array}$$

2.2 Pure IO

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$$\begin{array}{lll} \text{(IO)} & \max & \boldsymbol{c}'\boldsymbol{x} \\ & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & & \boldsymbol{x} \in Z_+^n \end{array}$$

Important special case: Binary Optimization

$$\begin{array}{lll} \text{(BO)} & \max & \boldsymbol{c}'\boldsymbol{x} \\ & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & & \boldsymbol{x} \in \{0,1\}^n \end{array}$$

2.3 LO

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$$\begin{array}{lll} \text{(LO)} & \max & \boldsymbol{c}'\boldsymbol{x} \\ & \text{s.t.} & \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{b} \\ & & \boldsymbol{y} \in R_+^n \end{array}$$

3 Modeling with Binary Variables

3.1 Binary Choice

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$$x \in \begin{cases} 1, & \text{if event occurs} \\ 0, & \text{otherwise} \end{cases}$$

Example 1: IO formulation of the knapsack problem

n : projects, total budget b

a_j : cost of project j

c_j : value of project j

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$$x_j = \begin{cases} 1, & \text{if project } j \text{ is selected.} \\ 0, & \text{otherwise.} \end{cases}$$

$$
\begin{array}{ll}
\max & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} & \sum a_j x_j \le b \\
& x_j \in \{0,1\}
\end{array}
$$

3.2 Modeling relations

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- At most one event occurs

$$\sum_j x_j \le 1$$

- Neither or both events occur

$$x_2 - x_1 = 0$$

- If one event occurs then, another occurs

$$0 \le x_2 \le x_1$$

- If $x = 0$, then $y = 0$; if $x = 1$, then y is uncontrained

$$0 \le y \le Ux, \qquad x \in \{0,1\}$$

3.3 The assignment problem

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$$
\begin{array}{rl}
n & \text{people} \\
m & \text{jobs} \\
c_{ij}: & \text{cost of assigning person } j \text{ to job } i. \\
x_{ij} & = \begin{cases} 1 & \text{person } j \text{is assigned to job } i \\ 0 & \end{cases} \\
\min & \sum c_{ij} x_{ij} \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} = 1 \quad \text{each job is assigned} \\
& \sum_{i=1}^{m} x_{ij} \le 1 \quad \text{each person can do at most one job.} \\
& x_{ij} \in \{0,1\}
\end{array}
$$

3.4 Multiple optimal solutions

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- Generate all optimal solutions to a BOP.

$$
\begin{array}{ll}
\max & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\
& \boldsymbol{x} \in \{0,1\}^n
\end{array}
$$

- $\boldsymbol{x}^*$ optimal solution: $I_0 = \{j : x_j^* = 0\}$, $I_1 = \{j : x_j^* = 1\}$.

- Add constraint

$$\sum_{j \in I_0} x_j + \sum_{j \in I_1} (1 - x_j) \geq 1.$$

- Generate third best?
- Extensions to MIO?

4 What is a good formulation?

4.1 Facility Location

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- Data

$$\begin{array}{ll} N = \{1 \dots n\} & \text{potential facility locations} \\ I = \{1 \dots m\} & \text{set of clients} \\ c_j : & \text{cost of facility placed at } j \\ h_{ij} : & \text{cost of satisfying client } i \text{ from facility } j. \end{array}$$

- Decision variables

$$\begin{array}{lll} x_j & = & \begin{cases} 1, & \text{a facility is placed at location } j \\ 0, & \text{otherwise} \end{cases} \\ y_{ij} & = & \text{fraction of demand of client } i \\ & & \text{satisfied by facility } j. \end{array}$$

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$$\begin{array}{rl} IZ_1 = \min & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \sum_{j=1}^{n} h_{ij} y_{ij} \\ \text{s.t.} & \sum_{j=1}^{n} y_{ij} = 1 \\ & y_{ij} \leq x_j \\ & x_j \in \{0, 1\}, 0 \leq y_{ij} \leq 1. \end{array}$$

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Consider an alternative formulation.

$$\begin{array}{rl} IZ_2 = \min & \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \sum_{j=1}^{n} h_{ij} y_{ij} \\ \text{s.t.} & \sum_{j=1}^{n} y_{ij} = 1 \\ & \sum_{i=1}^{m} y_{ij} \leq m \cdot x_j \\ & x_j \in \{0, 1\}, 0 \leq y_{ij} \leq 1. \end{array}$$

Are both valid?
Which one is preferable?

4.2 Observations

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- $IZ_1 = IZ_2$, since the integer points both formulations define are the same.
-

$$P_1 = \{(\boldsymbol{x}, \boldsymbol{y}) : \sum_{j=1}^{n} y_{ij} = 1, y_{ij} \leq x_j, \begin{array}{l} 0 \leq x_j \leq 1 \\ 0 \leq y_{ij} \leq 1 \end{array} \Big\}$$

$$P_2 = \{(\boldsymbol{x}, \boldsymbol{y}) : \sum_{j=1}^{n} y_{ij} = 1, \sum_{i=1}^{m} y_{ij} \leq m \cdot x_j, \begin{array}{l} 0 \leq x_j \leq 1 \\ 0 \leq y_{ij} \leq 1 \end{array} \Big\}$$

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- Let

$$Z_1 = \begin{array}{c} \min \boldsymbol{cx} + \boldsymbol{hy}, \\ (\boldsymbol{x}, \boldsymbol{y}) \in P_1 \end{array} \qquad Z_2 = \begin{array}{c} \min \boldsymbol{cx} + \boldsymbol{hy} \\ (\boldsymbol{x}, \boldsymbol{y}) \in P_2 \end{array}$$

- $Z_2 \leq Z_1 \leq IZ_1 = IZ_2$

4.3 Implications

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- Finding $IZ_1 (= IZ_2)$ is difficult.
- Solving to find Z_1, Z_2 is a LOP. Since Z_1 is closer to IZ_1 several methods (branch and bound) would work better (actually much better).
- Suppose that if we solve $\min \boldsymbol{cx} + \boldsymbol{hy}, (\boldsymbol{x}, \boldsymbol{y}) \in P_1$ we find an integral solution. Have we solved the facility location problem?

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- Formulation 1 is better than Formulation 2. (Despite the fact that 1 has a larger number of constraints than 2.)
- What is then the criterion?

4.4 Ideal Formulations

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- Let P be a linear relaxation for a problem
- Let

$$H = \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x} \in \{0,1\}^n\} \cap P$$

- Consider Convex Hull (H)

$$= \{\boldsymbol{x} : \boldsymbol{x} = \sum_i \lambda_i x^i, \sum_i \lambda_i = 1, \lambda_i \geq 0, x^i \in H\}$$

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- The extreme points of $CH(H)$ have $\{0,1\}$ coordinates.
- So, if we know $CH(H)$ explicitly, then by solving $\min \boldsymbol{cx} + \boldsymbol{hy}, (\boldsymbol{x}, \boldsymbol{y}) \in CH(H)$ we solve the problem.
- Message: Quality of formulation is judged by closeness to $CH(H)$.

$$CH(H) \subseteq P_1 \subseteq P_2$$

5 Minimum Spanning Tree (MST)

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- How do telephone companies bill you?
- It used to be that rate/minute: Boston → LA proportional to distance in MST
- Other applications: Telecommunications, Transportation (good lower bound for TSP)

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- Given a graph $G = (V, E)$ undirected and Costs c_e, $e \in E$.
- Find a tree of minimum cost spanning all the nodes.
- Decision variables $x_e = \begin{cases} 1, & \text{if edge } e \text{ is included in the tree} \\ 0, & \text{otherwise} \end{cases}$

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- The tree should be connected. How can you model this requirement?
- Let S be a set of vertices. Then S and $V \setminus S$ should be connected
- Let $\delta(S) = \left\{e = (i,j) \in E : \begin{array}{l} i \in S \\ j \in V \setminus S \end{array}\right\}$
- Then,

$$\sum_{e \in \delta(S)} x_e \geq 1$$

- What is the number of edges in a tree?
- Then, $\sum_{e \in E} x_e = n - 1$

5.1 Formulation

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$$IZ_{MST} = \min \sum_{e \in E} c_e x_e$$

$$H \quad \begin{cases} \sum_{e \in \delta(S)} x_e \geq 1 & \forall\, S \subseteq V, S \neq \emptyset, V \\ \sum_{e \in E} x_e = n - 1 \\ x_e \in \{0, 1\}. \end{cases}$$

Is this a good formulation?

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$$P_{cut} = \{\boldsymbol{x} \in R^{|E|} :\ 0 \leq \boldsymbol{x} \leq \boldsymbol{e},$$

$$\sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in \delta(S)} x_e \geq 1\ \forall\, S \subseteq V, S \neq \emptyset, V\}$$

Is P_{cut} the $CH(H)$?

5.2 What is $CH(H)$?

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Let

$$P_{sub} = \{\boldsymbol{x} \in R^{|E|} : \sum_{e \in E} x_e = n - 1$$

$$\sum_{e \in E(S)} x_e \leq |S| - 1\ \forall\, S \subseteq V,\ S \neq \emptyset, V\}$$

$$E(S) = \left\{ e = (i, j) : \begin{array}{c} i \in S \\ j \in S \end{array} \right\}$$

Why is this a valid IO formulation?

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- Theorem: $P_{sub} = CH(H)$.
- $\Rightarrow P_{sub}$ is the best possible formulation.
- MESSAGE: Good formulations can have an exponential number of constraints.

6 The Traveling Salesman Problem

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Given $G = (V, E)$ an undirected graph. $V = \{1, \ldots, n\}$, costs $c_e\ \forall\, e \in E$. Find a tour that minimizes total length.

6.1 Formulation I

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$$x_e = \begin{cases} 1, & \text{if edge } e \text{ is included in the tour.} \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{array}{ll} \min & \sum_{e \in E} c_e x_e \\ \text{s.t.} & \sum_{e \in \delta(S)} x_e \geq 2, \quad S \subseteq E \\ & \sum_{e \in \delta(i)} x_e = 2, \quad i \in V \\ & x_e \in \{0,1\} \end{array}$$

6.2 Formulation II

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$$\begin{array}{ll} \min & \sum c_e x_e \\ \text{s.t.} & \sum_{e \in E(S)} x_e \leq |S| - 1, \quad S \subseteq E \\ & \sum_{e \in \delta(i)} x_e = 2, \quad i \in V \\ & x_e \in \{0,1\} \end{array}$$

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$$\begin{array}{rl} P_{cut}^{TSP} = & \{x \in R^{|E|}; \sum_{e \in \delta(S)} x_e \geq 2, \sum_{e \in \delta(i)} x_e = 2 \\ & 0 \leq x_e \leq 1\} \\ P_{sub}^{TSP} = & \{x \in R^{|E|}; \sum_{e \in \delta(i)} x_e = 2 \\ & \sum_{e \in \delta(S)} x_e \leq |S| - 1 \\ & 0 \leq x_e \leq 1\} \end{array}$$

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- Theorem: $P_{cut}^{TSP} = P_{sub}^{TSP} \not\supseteq CH(H)$
- Nobody knows $CH(H)$ for the TSP

7 Minimum Matching

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- Given $G = (V, E); c_e$ costs on $e \in E$. Find a matching of minimum cost.
- Formulation:

$$\begin{array}{ll} \min & \sum c_e x_e \\ \text{s.t.} & \sum_{e \in \delta(i)} x_e = 1, \quad i \in V \\ & x_e \in \{0,1\} \end{array}$$

- Is the linear relaxation $CH(H)$?

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Let

$$
\begin{aligned}
P_{MAT} = \ & \{x \in R^{|E|} : \sum_{e \in \delta(i)} x_e = 1 \\
& \sum_{e \in \delta(S)} x_e \geq 1 \quad |S| = 2k+1, S \neq \emptyset \\
& x_e \geq 0\}
\end{aligned}
$$

Theorem: $P_{MAT} = CH(H)$

8 Observations

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- For MST, Matching there are efficient algorithms. $CH(H)$ is known.
- For TSP $\nexists$ efficient algorithm. TSP is an $NP-hard$ problem. $CH(H)$ is not known.
- Conjuecture: The convex hull of problems that are polynomially solvable are explicitly known.

9 Summary

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1. An IO formulation is better than another one if the polyhedra of their linear relaxations are closer to the convex hull of the IO.
2. A good formulation may have an exponential number of constraints.
3. Conjecture: Formulations characterize the complexity of problems. If a problem is solvable in polynomial time, then the convex hull of solutions is known.