2.098/6.255/15.093J Optimization Methods, Fall 2005
(Brief) Solutions to Final Exam, Fall 2003

1.

1. False. The problem of *minimizing* a convex, piecewise linear function over a polyhedron can be formulated as a LP.
2. True. The dual of the problem is $\max\{0 : p \leq 1\}$. $p = 1$ is nondegenerate, for example.
3. False. Consider $\min\{-x_1 - x_2 : x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\}$.
4. False. Take the primal-dual pair in part 2 of this question, for example.
5. False. Barrier interior-point methods are unaffected by degeneracy; see BT p. 439.
6. True. KKT conditions hold for a local minimum under the linearly independent constraint qualification condition (LICQ).
7. False. Barrier interior-point methods find an interior point of the face of optimal solutions. See BT p. 537 and p. 544 for a discussion on the numerical behavior of the simplex and interior point methods.
8. True. BT Theorem 7.5.
9. True. Lecture 18, Slides 40-50.
10. True. Recall the zig-zag phenomenon shown in lecture.

2.

(a) Proof by contradiction. Assume that f is strictly convex. Suppose all optimal solutions are not extreme points of P. Consider an arbitrary optimal solution, $x^* = (x_1^*, \ldots, x_n^*)$. Since x^* is not an extreme point, $x^* = \lambda y + (1-\lambda)z$ for some $y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n) \in P$ and $\lambda \in [0, 1]$. Therefore,

$$\lambda \sum_{j=1}^{n} f(y_i) + (1-\lambda) \sum_{j=1}^{n} f(z_i) < \sum_{j=1}^{n} f(x_i^*),$$

so either y or z must produce a lower value than x^*. This is a contradiction.
If f is not strictly convex, you can repeat the above argument in conjunction with an argument like in the proof of BT Theorem 2.6 ((b) $\Rightarrow$ (a)) to show that $\sum_{k=1}^{p} \lambda_i \sum_{i=1}^{n} f(x_i^k) \leq \sum_{i=1}^{n} f(x_i^*)$ where x^k is an extreme point for some $k = 1, \ldots, p$.

(b) The problem we are concerned with is

$$\begin{array}{rl} \text{minimize} & \sum_{j=1}^{n} f(x_j) \\ \text{subject to} & Ax = b \\ & x_j \in \{0, 1\} \end{array}$$

Let $c = f(1)$ and $d = f(0)$. Since $x_j \in \{0, 1\}$, $f(x_j) = d + (c-d)x_j$. Therefore, the objective function can be written as

$$\sum_{j=1}^{n} f(x_j) = \sum_{j=1}^{n} (d + (c-d)x_j) = nd + (c-d) \sum_{j=1}^{n} x_j,$$

which is linear in x.

3. Without loss of generality, assume Q and Σ are symmetric, since they only appear in quadratic forms.

(a) KKT conditions: there exists a multiplier $u \geq 0$ such that $(c + Qx) + u(d + \Sigma x) = 0$, and $u(d'x + \frac{1}{2}x'\Sigma x - a) = 0$.

(b) Use Newton's method to solve the system of equations prescribed by the KKT conditions.

(c) An equivalent optimization problem is

$$\begin{array}{rl} \text{minimize} & \theta \\ \text{subject to} & c'x + \frac{1}{2}x'Qx \leq \theta \\ & d'x + \frac{1}{2}x'\Sigma x \leq a \end{array}$$

Since Q is symmetric psd, we can write $Q = Q^{1/2}Q^{1/2}$ for some symmetric matrix $Q^{1/2}$. Similarly, $\Sigma = \Sigma^{1/2}\Sigma^{1/2}$ for some symmetric matrix $\Sigma^{1/2}$. Therefore, by the Schur complement lemma

$$(\theta - c'x) - \frac{1}{2}(Q^{1/2}x)'(Q^{1/2}x) \geq 0 \quad \Leftrightarrow \quad \begin{pmatrix} I & \frac{1}{\sqrt{2}}(Q^{1/2}x) \\ \frac{1}{\sqrt{2}}(Q^{1/2}x)' & \theta - c'x \end{pmatrix} \succcurlyeq 0.$$

Similarly,

$$(a - d'x) - \frac{1}{2}(\Sigma^{1/2}x)'(\Sigma^{1/2}x) \geq 0 \quad \Leftrightarrow \quad \begin{pmatrix} I & \frac{1}{\sqrt{2}}(\Sigma^{1/2}x) \\ \frac{1}{\sqrt{2}}(\Sigma^{1/2}x)' & a - d'x \end{pmatrix} \succcurlyeq 0.$$

So we can recast the given optimization problem as the following semidefinite programming problem:

$$\begin{array}{rl} \text{minimize} & \theta \\ \text{subject to} & \begin{pmatrix} I & \frac{1}{\sqrt{2}}(Q^{1/2}x) \\ \frac{1}{\sqrt{2}}(Q^{1/2}x)' & \theta - c'x \end{pmatrix} \succcurlyeq 0 \\ & \begin{pmatrix} I & \frac{1}{\sqrt{2}}(\Sigma^{1/2}x) \\ \frac{1}{\sqrt{2}}(\Sigma^{1/2}x)' & a - d'x \end{pmatrix} \succcurlyeq 0 \end{array}$$

Note that in the above formulation that the decision variables are θ and x, and they appear linearly in the matrix constraints.

4.

(a) A possible LP formulation is:

$$\begin{array}{rll} z^* = \text{maximize} & \theta & \\ \text{subject to} & x_i'f \leq 1 & \forall i : a_i = 0 \\ & x_i'f \geq 1 + \theta & \forall i : a_i = 1 \end{array}$$

where $f \in \mathbb{R}^n$ and θ are decision variables. If $z^* \leq 0$, then a separating hyperplane does not exist; if $z^* > 0$, then the optimal solution f^* defines a separating hyperplane.

(b) A possible integer linear programming formulation is:

$$\begin{array}{lrcll} \text{minimize} & \sum_{i=1}^m w_i & + & \sum_{i=1}^m z_i & \\ \text{subject to} & x_i' f & \leq & 1 + M u_i & i = 1,\ldots,m \\ & x_i' f & \geq & (1+\epsilon) - M(1-u_i) & i = 1,\ldots,m \\ & w_i & \geq & (y_i - \beta_1' x_i) - M u_i & i = 1,\ldots,m \\ & w_i & \geq & -(y_i - \beta_1' x_i) - M u_i & i = 1,\ldots,m \\ & w_i & \leq & M(1-u_i) & i = 1,\ldots,m \\ & z_i & \geq & (y_i - \beta_2' x_i) - M(1-u_i) & i = 1,\ldots,m \\ & z_i & \geq & -(y_i - \beta_2' x_i) - M(1-u_i) & i = 1,\ldots,m \\ & z_i & \leq & M u_i & i = 1,\ldots,m \\ & u_i & \in & \{0,1\} & i = 1,\ldots,m \end{array}$$

where $w, z \in \mathbb{R}$, $\beta_1, \beta_2, f \in \mathbb{R}^n$, $u \in \mathbb{Z}^n$ are decision variables, M is some "very large" constant, and ϵ is some "very small" constant. Note that $u_i = 0$ implies $x_i' f \leq 1$, $w_i \geq |y_i - \beta_1' x_i|$, and $z_i = 0$. Also note that $u_i = 1$ implies $x_i' f \geq (1+\epsilon) > 1$, $w_i = 0$, and $z_i \geq |y_i - \beta_2' x_i|$.

5.

(a) We can compute the value of Z_1 by subgradient methods, as indicated in BT pp. 502-507. Let $n = 2$, $a_1' = (2,3)$, $a_2' = (3,2)$, $b_1 = 2$, $b_2 = 3$. In this instance, neither of the equalities in BT Corollary 11.1 hold, so we can only say $Z_{LP} \leq Z_1 \leq Z_{IP}$.

(b) We consider one variable at a time, in the order $x_1, x_2, \ldots, x_n$. Accordingly, we define our time periods to be $k = 1, \ldots, n$. Define the states to be the ordered pairs (d, f), where d represents the running total of the LHS of the first constraint, and f represents the running total of the LHS of the second constraint. The actions available at time period k correspond to setting the value of x_k to 0 or 1. The cost-to-go function is defined as follows:

$$\begin{array}{rcll} J_k(d,f) & = & \text{minimize} & \sum_{i=k}^n c_i x_i \\ & & \text{subject to} & d + \sum_{i=k}^n a_{1i} x_i \geq b_1 \\ & & & f + \sum_{i=k}^n a_{2i} x_i \geq b_2 \\ & & & x_i \in \{0,1\},\ i = k, \ldots, n \end{array}$$

We can solve for the value we desire, $J_1(0,0)$, using the following recursion

$$J_k(d_k, f_k) = \min\{\underbrace{c_k + J_{k+1}(d_k + a_{1k}, f_k + a_{2k})}_{x_k = 1}, \underbrace{J_{k+1}(d_k, f_k)}_{x_k = 0}\}$$

with the following boundary conditions:

$$\begin{array}{rcll} J_n(d,f) & = & \text{minimize} & c_n x_n \\ & & \text{subject to} & d + a_{1n} x_n \geq b_1 \\ & & & f + a_{2n} x_n \geq b_2 \\ & & & x_n \in \{0,1\} \end{array}$$

$$\Rightarrow J_n(d,f) = \begin{cases} 0 & \text{if } d \geq b_1 \text{ and } f \geq b_2 \\ c_n & \text{if } d < b_1 \leq d + a_{1n} \text{ or } f < b_2 \leq f + a_{2n} \\ \infty & \text{otherwise.} \end{cases}$$

Note that $0 \leq d \leq \sum_{i=1}^n a_{1i}$ and $0 \leq f \leq \sum_{i=1}^n a_{2i}$. If a_1 and a_2 are integral, then the state space is finite, of cardinality $(\sum_{i=1}^n a_{1i} + 1)(\sum_{i=1}^n a_{2i} + 1)$. If a_1 and a_2 are not integral, then the state space becomes uncountable.

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