# 15.083J/6.859J Integer Optimization

Lecture 6: Ideal formulations II

# 1 Outline

Slide 1

- Randomized rounding methods

# 2 Randomized rounding

Slide 2

- Solve $\boldsymbol{c}'\boldsymbol{x}$ subject to $\boldsymbol{x} \in P$ for arbitrary $\boldsymbol{c}$.
- $\boldsymbol{x}^*$ be optimal solution.
- From $\boldsymbol{x}^*$ create a new random integer solution $\boldsymbol{x}$, feasible in $P$: $\mathrm{E}[\boldsymbol{c}'\boldsymbol{x}] = Z_{\mathrm{LP}} = \boldsymbol{c}'\boldsymbol{x}^*$.
- $Z_{\mathrm{LP}} \leq Z_{\mathrm{IP}} \leq \mathrm{E}[Z_{\mathrm{H}}] = Z_{\mathrm{LP}}$.
- Hence, $P$ integral.

## 2.1 Minimum $s - t$ cut

Slide 3

$$\begin{aligned}
\text{minimize} \quad & \sum_{\{u,v\}\in E} c_{uv} x_{uv} \\
\text{subject to} \quad & x_{uv} \geq y_u - y_v, && \{u,v\} \in E, \\
& x_{uv} \geq y_v - y_u, && \{u,v\} \in E, \\
& y_s = 1, \\
& y_t = 0, \\
& y_u, x_{uv} \in \{0,1\}.
\end{aligned}$$

### 2.1.1 Algorithm

Slide 4

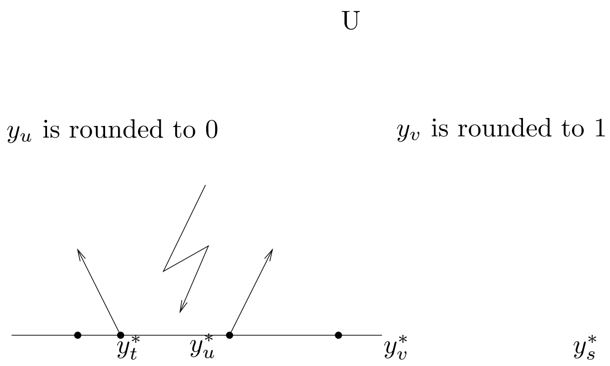
- Solve linear relaxation. Position the nodes in the interval $(0,1)$ according to the value of $y_u^*$.
- Generate a random variable $U$ uniformly in the interval $[0,1]$.
- Round all nodes $u$ with $y_u^* \leq U$ to $y_u = 0$, and all nodes $u$ with $y_u^* > U$ to $y_u = 1$. Set $x_{uv} = |y_u - y_v|$ for all $\{u,v\} \in E$.

## 2.2 Theorem

Slide 5

For every nonnegative cost vector $\boldsymbol{c}$,

$$\mathrm{E}[Z_{\mathrm{H}}] = Z_{\mathrm{IP}} = Z_{\mathrm{LP}}.$$




Proof:

$$
\begin{aligned}
Z_{\mathrm{IP}} &\leq \mathrm{E}[Z_{\mathrm{H}}] = \mathrm{E}\left[\sum_{\{u,v\}\in E} c_{uv} x_{uv}\right] \\
&= \sum_{\{u,v\}\in E} c_{uv} \mathrm{P}\Bigg(\min\left(y_u^*, y_v^*\right) \leq U < \max\left(y_u^*, y_v^*\right)\Bigg) \\
&= \sum_{\{u,v\}\in E} c_{uv} |y_u^* - y_v^*| \\
&= Z_{\mathrm{LP}} \leq Z_{\mathrm{IP}}
\end{aligned}
$$

## 2.3 Stable matching

SLIDE 6

- $n$ men $\{m_1, \ldots, m_n\}$ and $n$ women $\{w_1, \ldots, w_n\}$, with each person having a list of strict preference order.
- Find a stable perfect matching $M$ of the men to women:
- There does not exist a man $m$ and a woman $w$ who are not matched under $M$, but prefer each other to their assigned mates under $M$.

### 2.3.1 Formulation

SLIDE 7

- $w_1 >_m w_2$ if man $m$ prefers $w_1$ to $w_2$.
- $m_1 >_w m_2$ if woman $w$ prefers $m_1$ to $m_2$.
- Decision variables

$$
x_{ij} = \begin{cases} 1, & \text{if } m_i \text{ is matched to } w_j, \\ 0, & \text{otherwise.} \end{cases}
$$

- $N = \{1, \ldots, n\}$

$$\begin{aligned}
\sum_{j=1}^{n} x_{ij} &= 1, && i \in N, \\
\sum_{i=1}^{n} x_{ij} &= 1, && j \in N, \\
x_{ij} &\in \{0, 1\}, && i, j \in N, \\
x_{ij} + \sum_{\{k \mid w_k <_{m_i} w_j\}} x_{ik} + \sum_{\{k \mid m_k <_{w_j} m_i\}} x_{kj} &\leq 1, && i, j \in N.
\end{aligned}$$

### 2.3.2 Proposition

Slide 8

$\boldsymbol{x} \in P_{\mathrm{SM}}$. If $x_{ij} > 0$, then

$$x_{ij} + \sum_{\{k \mid w_k <_{m_i} w_j\}} x_{ik} + \sum_{\{k \mid m_k <_{w_j} m_i\}} x_{kj} = 1.$$

### 2.3.3 Proof

Slide 9

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \\
\text{s.t.} \quad & \boldsymbol{x} \in P_{\mathrm{SM}}
\end{aligned}$$

Dual

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j - \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \\
\text{s.t.} \quad & \alpha_i + \beta_j - \sum_{\{k \mid w_k >_{m_i} w_j\}} \gamma_{ik} - \sum_{\{k \mid m_k >_{w_j} m_i\}} \gamma_{kj} \leq 1, \quad i, j \in N, \\
& \gamma_{ij} \geq 0.
\end{aligned}$$

$\boldsymbol{x} \in P_{\mathrm{SM}}$. Set

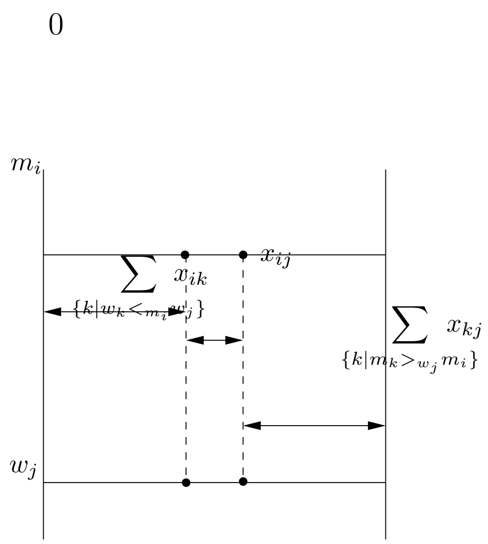
$$\alpha_i = \sum_{j=1}^{n} \gamma_{ij}, \ \beta_j = \sum_{i=1}^{n} \gamma_{ij} \text{ and } \gamma_{ij} = x_{ij} \text{ for all } i, j \in N.$$

- Dual:

  $$\gamma_{ij} + \sum_{\{k \mid w_k <_{m_i} w_j\}} \gamma_{ik} + \sum_{\{k \mid m_k <_{w_j} m_i\}} \gamma_{kj} \leq 1, \qquad \forall\ i, j \in N,$$

  feasible if $\gamma_{ij} = x_{ij}$ and $\boldsymbol{x} \in P_{\mathrm{SM}}$.
- Objective

  $$\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j - \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}.$$
- Complementary slackness of optimal primal and dual solutions.




## 2.4 Key Theorem

SLIDE 10

$P_{\mathrm{SM}} = \mathrm{conv}(S)$.

### 2.4.1 Randomization

SLIDE 11

- Generate a random number $U$ uniformly in [0,1].
- Match $m_i$ to $w_j$ if $x_{ij} > 0$ and in the row corresponding to $m_i$, $U$ lies in the interval spanned by $x_{ij}$ in $[0, 1]$. Accordingly, match $w_j$ to $m_i$ if in the row corresponding to $w_j$, $U$ lies in the interval spanned by $x_{ij}$ in $[0, 1]$.
- Key property: $x_{ij} > 0$, then the intervals spanned by $x_{ij}$ in rows corresponding to $m_i$ and $w_j$ coincide in $[0, 1]$.

SLIDE 12

- The matching is stable: $w_k$ who is preferred by $m_i$ to his mate $w_j$ under the assignment, i.e., the interval spanned by $x_{ik}$ is on the right of the interval spanned by $x_{ij}$ in the row corresponding to $m_i$, is assigned a mate whom she strictly prefers to $m_i$, since in the row corresponding to $w_k$ the random number $U$ lies strictly to the left of the interval $x_{ik}$.
- $x_{ij}^U = 1$ if $m_i$ and $w_j$ are matched.

$$\mathrm{E}[x_{ij}^U] = \mathrm{P}(U \text{ lies in the interval spanned by } x_{ij}) = x_{ij}.$$

- $x_{ij} = \int_0^1 x_{ij}^u du$: $\boldsymbol{x}$ can be written as a convex combination of stable matchings $\boldsymbol{x}^u$ as $u$ varies over the interval $[0, 1]$.

15.083J / 6.859J Integer Programming and Combinatorial Optimization
Fall 2009