IP Formulation guide (abbreviated)

15.053 and 15.058

Spring 2013

- For use on the quiz on April 4, 2013
  and for the midterm on April 18, 2013.
- DO NOT WRITE ON THIS DOCUMENT
- This document will be provided with the quiz and midterm.
- Hand in this guide at the end of the quiz and midterm.

1. Packing, covering, and partitioning problems.

There is a collection of sets $S_1, \ldots, S_n$ with $S_i \subseteq \{1, 2, 3, \ldots, m\}$ for $i = 1$ to n.
Associated with each set S_i is a cost c_i.

$$\text{Let } a_{ij} = \begin{cases} 1 & \text{if } i \in S_j \\ 0 & \text{otherwise.} \end{cases} \qquad \text{Let } x_j = \begin{cases} 1 & \text{if set } S_j \text{ is selected} \\ 0 & \text{otherwise.} \end{cases}$$

The set packing problem is the problem of selecting the maximum cost subcollection of sets, no two of which share a common element. *The set covering problem* is the problem of selecting the minimum cost subcollection of sets, so that each element $i \in \{1, 2, \ldots, m\}$ is in one of the sets.

$$\begin{aligned} &\text{Maximize} && \textstyle\sum_{j=1}^{n} c_j x_j \\ &\text{subject to} && \textstyle\sum_{j=1}^{n} a_{ij} x_j \le 1 \quad \text{for each } i \in \{1,\ldots,m\} \\ &&& x_j \in \{0,1\} \qquad \text{for each } j \in \{1,\ldots,n\}. \end{aligned} \qquad \textit{Set Packing Problem}$$

$$\begin{aligned} &\text{Minimize} && \textstyle\sum_{j=1}^{n} c_j x_j \\ &\text{subject to} && \textstyle\sum_{j=1}^{n} a_{ij} x_j \ge 1 \quad \text{for each } i \in \{1,\ldots,m\} \\ &&& x_j \in \{0,1\} \qquad \text{for each } j \in \{1,\ldots,n\}. \end{aligned} \qquad \textit{Set Covering Problem}$$

Section 2. Modular arithmetic.

$x \equiv a \pmod b$ is the same as: there is an integer q such that $a + qb = x$. integer.

Constraint.	IP Constraint
x is odd.	$x - 2w = 1$, $w \ge 0$, w is integral.
x is even.	$x - 2w = 0$, $w \ge 0$, w is integral.
$x \equiv a \pmod b$	$x - bw = a$, $w \ge 0$, w is integral.

Table 1. Modular arithmetic formulations.

3. Restricting a variable to take on one of several values.

$x \in \{4, 8, 13\}$	$x = 4 w_1 + 8 w_2 + 13 w_3$; $w_1 + w_2 + w_3 = 1$ $w_i \in \{0, 1\}$ for $i = 1$ to 3.
$x \in \{0, 4, 8, 13\}$	$x = 4 w_1 + 8 w_2 + 13 w_3$; $w_1 + w_2 + w_3 \le 1$ $w_i \in \{0, 1\}$ for $i = 1$ to 3.

Logical Constraints.	**IP Constraint**
If item *i* is selected, then item *j* is also selected.	$x_i - x_j \le 0$
Either item *i* is selected or item *j* is selected, but not both.	$x_i + x_j = 1$
Item *i* is selected or item *j* is selected or both.	$x_i + x_j \ge 1$
If item *i* is selected, then item *j* is not selected.	$x_i + x_j \le 1$
If item *i* is not selected, then item *j* is not selected.	$-x_i + x_j \le 0$
At most one of items *i*, *j*, and *k* are selected.	$x_i + x_j + x_k \le 1$
Exactly one of items *i*, *j*, and *k* are selected.	$x_i + x_j + x_k = 1$
At least one of items *i*, *j* and *k* are selected.	$x_i + x_j + x_k \ge 1$

Table 2. Simple constraints involving two or three binary variables.

4. Logical constraints and the big M method.

In the following, we assume that all integer variables are bounded from above. We also assume that all variables are required to be integer valued. We let M denote some very large integer.

Logical Constraints using big M method.	**IP Constraint**
$x_1 + 4x_2 - 2x_4 \ge 7$ **or** $3x_1 - 5x_2 \le 12$	$x_1 + 4x_2 - 2x_4 \ge 7 - Mw$ $3x_1 - 5x_2 \le 12 + M(1-w)$ $w \in \{0,1\}$
If $x_1 + 4x_2 - 2x_4 \ge 7$ **then** $3x_1 - 5x_2 \le 12$	$x_1 + 4x_2 - 2x_4 \le 6 + Mw$ $3x_1 - 5x_2 \le 12 + M(1-w)$ $w \in \{0,1\}$
At least two of the following constraints are satisfied: $x_1 + 4x_2 + 2x_4 \ge 7$; $3x_1 - 5x_2 \le 12$; $2x_2 + x_3 \ge 6$;	$x_1 + 4x_2 + 2x_4 \ge 7 - M(1-w_1)$ $3x_1 - 5x_2 \le 12 + M(1-w_2)$ $2x_2 + x_3 \ge 6 - M(1-w_3)$ $w_1 + w_2 + w_3 \ge 2$ $w_i \in \{0,1\}$ for $i = 1$ to 3.
At most one of the following constraints are satisfied: $x_1 + 4x_2 + 2x_4 \ge 7$; $3x_1 - 5x_2 \le 12$; $2x_2 + x_3 \ge 6$;	This is equivalent to : **at least two** of the following constraints are satisfied. $x_1 + 4x_2 + 2x_4 \le 6$; $3x_1 - 5x_2 \ge 13$; $2x_2 + x_3 \le 5$;

3

5. Fixed costs

Here we consider an integer program in which there are fixed costs on variables. For example, consider the following integer program:

Formulation using fixed costs

$$\begin{array}{ll} \text{Maximize} & f(x_1) + f_2(x_2) + f_3(x_3) \\ \text{subject to} & 2x_1 + 4x_2 + 5x_3 \le 100 \\ & x_1 + x_2 + x_3 \le 30 \\ & 10x_1 + 5x_2 + 2x_3 \le 204 \\ & x_i \ge 0 \text{ and integer for } i = 1 \text{ to } 3. \end{array}$$

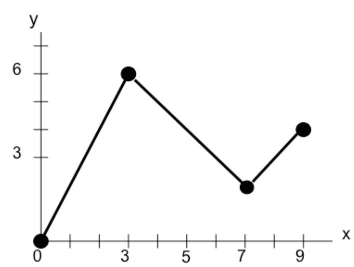
IP Formulation

$$\begin{array}{ll} \text{Maximize} & 52x_1 - 500w_1 + 30x_2 - 400w_2 + 20x_3 - 300w_3 \\ \text{subject to} & 2x_1 + 4x_2 + 5x_3 \le 100 \\ & x_1 + x_2 + x_3 \le 30 \\ & 10x_1 + 5x_2 + 2x_3 \le 204 \\ & x_i \le Mw_i \quad \text{for } i = 1 \text{ to } 3 \\ & x_i \ge 0 \text{ and integer for } i = 1 \text{ to } 3. \end{array}$$

where $f_1(x_1) = \begin{cases} 52x_1 - 500 & \text{if } x_1 \ge 1 \\ 0 & \text{if } x_1 = 0 \end{cases}$, $f_2(x_2) = \begin{cases} 30x_2 - 400 & \text{if } x_2 \ge 1 \\ 0 & \text{if } x_2 = 0 \end{cases}$, and $f_3(x_3) = \begin{cases} 20x_3 - 300 & \text{if } x_3 \ge 1 \\ 0 & \text{if } x_3 = 0 \end{cases}$.

Section 6. Piecewise linear functions.

Integer programming can be used to model functions that are piecewise linear. For example, consider the following function.



$$y = \begin{cases} 2x & \text{if } 0 \le x \le 3 \\ 9 - x & \text{if } 4 \le x \le 7 \\ -5 + x & \text{if } 8 \le x \le 9. \end{cases}$$

One can model *y* in several different ways. Here is one of them. We first define two new variables for every piece of the curve.

Definitions of the variables

$$w_1 = \begin{cases} 1 & \text{if } 0 \le x \le 3 \\ 0 & \text{otherwise.} \end{cases} \qquad x_1 = \begin{cases} x & \text{if } 0 \le x \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

$$w_2 = \begin{cases} 1 & \text{if } 4 \le x \le 7 \\ 0 & \text{otherwise.} \end{cases} \qquad x_2 = \begin{cases} x & \text{if } 4 \le x \le 7 \\ 0 & \text{otherwise.} \end{cases}$$

$$w_3 = \begin{cases} 1 & \text{if } 8 \le x \le 9 \\ 0 & \text{otherwise.} \end{cases} \qquad x_3 = \begin{cases} x & \text{if } 8 \le x \le 9 \\ 0 & \text{otherwise.} \end{cases}$$

IP formulation

$0 \le x_1 \le 3w_1$

$4w_2 \le x_2 \le 7w_2$

$8w_3 \le x_3 \le 9w_3$

$w_1 + w_2 + w_3 = 1$

$x = x_1 + x_2 + x_3$

$w_i \in \{0, 1\}$ for $i = 1$ to 3.