PROBLEM SET 11

1. Formation of a black hole

An objection to the Schwarzschild black hole solution described in class is that it is "eternal". In this problem, we will see how a Schwarzschild black hole can be formed from the collapse of a simple, non-singular physical object.

We will examine the collapse of a "star" composed of pressureless dust. We take the star to be spherical, of initial radius R_*, of mass M, and composed of an isotropic distribution of pressureless dust. By Birkhoff's theorem, the exterior of this star is simply described by the Schwarzschild metric:

$$ds^2_{r>R_*} = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2 d\Omega^2 .$$

Since the interior is spatially isotropic, it is perfectly described by the Robertson-Walker metric. Since the star clearly must collapse under its own self gravity, we use the closed RW metric to describe the interior:

$$ds^2_{r<R_*} = -d\tau^2 + a^2(\tau)R_0^2 \left(d\chi^2 + \sin^2\chi d\Omega^2\right) .$$

(We use different notation for time on the interior and the exterior: The exterior time t is Schwarzschild time, a convenient label very far away, but perhaps confusing in the strong field; the interior time τ denotes the proper time experienced by an element of dust inside the star.)

(a) The evolution of the scale factor for a closed RW line element turns out to have a simple, closed form solution. Show that the parametric solution

$$a = \frac{a_{\max}}{2}(1 + \cos\eta) ,$$

$$\tau = \frac{a_{\max}R_0}{2}(\eta + \sin\eta) ,$$

with $0 \leq \eta \leq \pi$, solves the Friedmann equations for $k = 1$ assuming ρ is given by pressureless dust matter. Since a always appears multiplied by the lengthscale R_0 in the line element, you may set $a_{\max} = 1$. Find a relationship between the initial density ρ_0 and the lengthscale R_0.

(b) This solution for the interior time coordinate τ is only good up to $\tau = \pi R_0/2$. What happens to the interior solution after that?

We now need to examine the star's surface from the point of view of the external coordinate system.

(c) Consider a purely radial "orbit" (i.e., a trajectory with no angular momentum: $L = 0$). For a given energy per unit mass E, find the radius R at which the radial velocity goes to zero.

We will use this solution to define the "orbital energy" of a dust element at the surface of the star as it begins to collapse.

(d) Using the radial geodesic equation for the Schwarzschild geometry and the relationship between E and R you found in (b), write down an integral for the proper time τ it takes for a fluid element at the star's surface to fall from its initial radius R_* to r. You should find the answer

$$\tau = -\int_{R_*}^{r} \frac{dr'}{\sqrt{2GM/r' - 2GM/R_*}} \, .$$

(The minus sign is due to the infalling motion of the surface.) By introducing the parameterization

$$r = \frac{R_*}{2}(1 + \cos\eta) \, , \qquad 0 \le \eta \le \pi \, ,$$

show that this integral can be evaluated to give

$$\tau = \sqrt{\frac{R_*^3}{8GM}}(\eta + \sin\eta) \, .$$

We now match the inner and the outer coordinate systems: We require that the star's circumference be the same in both the inner and the outer coordinate systems for all η, and we require that the two expressions for the proper time τ experienced by a fluid element on the star's surface be the same for all η.

(e) By enforcing these two conditions, determine the lengthscale R_0 and the Robertson-Walker radius of the star χ.

For the next part of the problem, assume that the star's initial radius is $R_* = 5GM$.

A Schwarzschild black hole's event horizon is a null surface: It is "generated" by null geodesics whose coordinate locations are $r = 2GM$ for all time. The event horizon of a black hole that forms in collapse is "generated" by the null geodesic that begins at the star's center and reaches the surface just as the surface passes through $r = 2GM$; at that point, by Birkhoff's theorem this horizon "generator" will remain at $r = 2GM$ for all time.

(f) Determine the time τ at which the horizon generator leaves the center of the star.

Hint: It is easiest to solve for this geodesic by noting that the parametric solution for the closed RW spacetime allows us to write

$$ds^2 = a(\eta)^2 R_0^2 \left(-d\eta^2 + d\chi^2 + \sin^2\chi \, d\Omega^2\right) \, .$$

An outward propagating null geodesic thus obeys $d\chi/d\eta = 1$. Using the value of χ you found for the star's surface, it should thus be easy to integrate backwards from the moment that the star's surface crosses $r = 2GM$ to determine the value of η at which the horizon generator leaves the star's center. You then just need to convert to τ.

(g) On a spacetime diagram, sketch the evolution of the star's surface and of the event horizon.

2. Consider a static, spherical star cluster in which all stars move in circular orbits. Ignore collisions between stars (i.e., approximate the stars as non-interacting dust). Adopt Schwarzschild-type coordinates with $r = 0$ at the center of the cluster and write the metric in the form

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2d\Omega^2 \ ,$$
$$\Phi = \Phi(r), \qquad \Lambda = \Lambda(r) \ .$$

(a) Find $e^{2\Lambda}$ and $d\Phi/dr$ in terms of $m = \int_0^r 4\pi\rho r^2 dr$, where $\rho = \rho(r)$ is the stars' mass density in the cluster. (We assume that there are enough stars that a continuum treatment is accurate.) Hint: Your final equations should be similar to the TOV equations but with one rather crucial difference.

(b) Define an appropriate effective potential $V_{\text{eff}}(r)$. Use it to determine the energy per unit mass $\hat{E}$ and angular momentum per unit mass $\hat{L}$ of a star in the cluster. Your answer should be expressed in terms of r, $m(r)$, and $\Phi(r)$. Determine the orbital frequency $\Omega \equiv d\phi/dt = (d\phi/d\tau)/(dt/d\tau)$.

(c) Use V_{eff} to analyze the stability of orbits of individual stars in the cluster. What local condition must $Gm(r)/r$ satisfy if all orbits at r are to be stable?

(d) Apply the above results to homogeneous cluster of total mass M and radius R. [Homogeneous means $\rho(r) = $ const, so $m(r) = M(r/R)^3$ for $r \leq R$; you will need to use this to solve for $\Phi(r)$ to complete this part of the problem.] Find the maximum value of GM/R if all orbits are to be stable.

(e) Find the cluster with maximal GM/R, compute the redshift of photons emitted from the cluster's surface, and from its center. When quasars were first discovered, their typical redshift was on the order of $z \sim 0.3$. Could a cluster of this type explain this redshift? Today, quasars are measured with redshifts as high as $z \simeq 6.5$. How well does the relativisic cluster hypothesis explain these quasars?

3. Numerical studies of black hole orbits

In lecture, we showed that the following equations govern the motion of a test body moving around a Schwarzschild black hole:

$$\left(\frac{dr}{d\tau}\right)^2 = \hat{E}^2 - V_{\text{eff}}(r), \qquad \text{where}$$

$$V_{\text{eff}}(r) = \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right);$$

$$\frac{d\phi}{d\tau} = \frac{\hat{L}}{r^2};$$

$$\frac{dt}{d\tau} = \frac{\hat{E}}{1 - 2GM/r}.$$

In this exercise, you will numerically integrate these equations to study some interesting black hole trajectories. Note, the equation for r involves a slightly troublesome singularity at turning points that can make numerical integration tricky: At a turning point, $dr/d\tau$ changes sign. Naively, you might imagine that this means you have to carefully change which root to take as you pass through these turning points. Instead, try taking an additional derivative of both sides and rearranging.

It is useful to define units in which $GM = 1$; this amounts to redefining $r \to r/GM$, $t \to t/GM$, $\tau \to \tau/GM$, $\hat{L} \to \hat{L}/GM$.

(a) With this choice, what are the basic units of time and length if $M = 10$ solar masses?

Now, construct examples of the following types of orbits:

(b) An unbound orbit — one which comes in from infinity, passes near the black hole, and zooms back out to infinity. How close can you get the orbit to come to the black hole without being captured?

(c) An "elliptical" bound orbit. You will rapidly discover that these orbits are not truly ellipses (as in Newtonian theory), but are at best *precessing* ellipses. Try to make an orbit whose innermost radius and outermost radius are as widely separated as possible.

(d) A stable circular orbit.

(e) An unstable circular orbit. How long can your integrator hold the unstable circular orbit at fixed r? It's entirely possible that your numerical integrator will hold this orbit for quite a long time; if that's the case, consider slightly perturbing the energy or angular momentum to see how it evolves. Can you drive the system into both outcomes of instability (plunge into the hole and escape to infinity)?

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