2 Cosmological Models with Idealized Matter

2.1 Model spaces: Construction

Spaces (and spacetimes) of high symmetry play a very important role in cosmological model-building, and as examples ("solvable models") of general relativity. The most important ones can be considered as different odd sorts of spheres, so we start with those

1. 3d sphere

$$\sum_{i=1}^{4} x_i^2 = R^2$$

- Spherical coordinates

$$x_1 = R\cos\left(\frac{x}{R}\right) \qquad x_3 = R\sin\left(\frac{x}{R}\right)\sin(\theta)\cos(\phi)$$
$$x_2 = R\sin\left(\frac{x}{R}\right) \qquad x_4 = R\sin\left(\frac{x}{R}\right)\sin(\theta)\sin(\phi)$$

$$dl^2 = \sum_i dx_i^2 \underbrace{=}_{\text{Exercise}} dx^2 + R^2\sin^2\left(\frac{x}{R}\right)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)$$

- Quasi-flat coordinates
Write $x_4^2 = R^2 - r^2$

$$dx_4 = \frac{rdr}{x_4} \qquad dx_4^2 = \frac{r^2dr^2}{R^2 - r^2}$$

$$\begin{aligned} dl^2 &= \sum_{i=1}^{3} dx_i^2 + dx_4^2 \\ &= dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) + \frac{r^2dr^2}{R^2 - r^2} \\ &= \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \qquad (1) \\ &= R^2\left(\frac{du^2}{1-u^2} + u^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right), u = \frac{r}{R} \end{aligned}$$

- Conformal coordinates It is often useful to write

$$ds^2 = f^2(x)ds^2_{\text{flat}}$$

if that is possible. (Penrose diagrams... later.) Starting from out previous form, we will have this if we use η in place of r such that

$$\begin{aligned} \frac{dr^2}{1 - \frac{r^2}{R^2}} &= f^2 d\eta^2 \\ r^2 &= f^2\eta^2 \\ \Rightarrow \frac{d\eta}{\eta} &= \frac{dr}{r\sqrt{1 - \frac{r^2}{R^2}}} \end{aligned}$$

leading to $\eta = \tan\left(\frac{u}{2}\right)$ with $\sin(u) = \frac{r}{R}$. $\Big($Write $r = R\sin(u)$; $\frac{dr}{r\sqrt{1-\frac{r^2}{R^2}}} = \frac{du}{\sin(u)} = d\left(\log\tan\left(\frac{u}{2}\right)\right) = \frac{d\eta}{\eta} = d\log(\eta)$. $\Big)$
or, after some algebra:

$$dl^2 = \frac{4R^2}{(1+\eta^2)^2}\left(d\eta^2 + \eta^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right)$$

$$\left(\quad f^2 = \frac{r^2}{\eta^2} \right.$$

$$\sin^2(u) = \frac{r^2}{R^2}$$

$$\left. 4\sin^2\left(\frac{u}{2}\right)\cos^2\left(\frac{u}{2}\right) = 4\left(\frac{\eta^2}{1+\eta^2}\right)\left(\frac{1}{1+\eta^2}\right)\right)$$

The sphere supports the symmetry $SO(4)$.

2. 3d hyperboloid (space of constant negative curvature)
   (figure)

$$x_0^2 - \sum_{i=1}^{3} x_i^2 = R^2$$

- Spherical coordinates

$$\begin{aligned} x_0 &= R\cosh\left(\frac{x}{R}\right) \\ x_1 &= R\sinh\left(\frac{x}{R}\right)\cos(\theta) \\ \ldots \\ dl^2 &= -dx_0^2 + d\mathbf{x}^2 \\ &= dx^2 + R^2\sinh^2\left(\frac{x}{R}\right)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \end{aligned}$$

- Quasi-flat coordinates

$$\begin{aligned} |\mathbf{x}| &= r \\ x_0^2 &= R^2 + r^2 \\ \ldots \\ dl^2 &= \frac{dr^2}{1+\frac{r^2}{R^2}} + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \\ &= R^2\left(\frac{du^2}{1+u^2} + u^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right) \end{aligned}$$

- Conformal coordinates

$$dl^2 = \frac{4}{(1-\eta^2)^2}\left(d\eta^2 + \eta^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right)$$

with $\eta = \tanh\left(\frac{u}{2}\right), \sinh(u) = \frac{r}{R}$

Supports symmetry $SO(3,1)$, *i.e.* "Lorentz" symmetry, acting purely spatially!

To bring this out, use

$$x_0 = \sqrt{r^2 + R^2}\cosh(\lambda) \qquad x_2 = r\cos(\phi)$$
$$x_1 = \sqrt{r^2 + R^2}\sinh(\lambda) \qquad x_3 = r\sin(\phi)$$

$$dl^2 = \frac{1}{1 + \frac{r^2}{R^2}} dr^2 + r^2 d\phi^2 + (r^2 + R^2) d\lambda^2$$

"Translations" $\lambda \to \lambda +$ constant, corrisponding to boosts in the original variables, leave this invariant.

3. de-Sitter spacetime

   (figure)

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -R^2$$

$$x_1 = R\cosh\left(\frac{x}{R}\right)\cos(\lambda)$$
$$x_2 = R\cosh\left(\frac{x}{R}\right)\sin(\lambda)\cos(\theta)$$
$$x_3 = R\cosh\left(\frac{x}{R}\right)\sin(\lambda)\sin(\theta)\cos(\phi)$$
$$x_4 = R\cosh\left(\frac{x}{R}\right)\sin(\lambda)\sin(\theta)\sin(\phi)$$
$$x_0 = R\sinh\left(\frac{x}{R}\right)$$

- Spherical coordinates

$$\begin{aligned} ds^2 &= dx_0^2 - \sum_i dx_i^2 \\ &= dx^2 - R^2\cosh^2\left(\frac{x}{R}\right)\Bigg(\underbrace{d\lambda^2 + \sin^2(\lambda)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)}_{\text{unit 3-sphere}}\Bigg) \end{aligned}$$

  (?): exponential expansion!; minimum radius; spheres

- Quasi-flat coordinates

$$x_0^2 = r^2 - R^2 \qquad dx_0^2 = \frac{r^2 dr^2}{r^2 - R^2}$$

$$ds^2 = \frac{dr^2}{\frac{r^2}{R^2} - 1} - r^2\left(d\lambda^2 + \sin^2(\lambda)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right)$$

- Light-front coordinates

  Seperate out planes $(x_2, x_3, x_4) = \mathbf{x}_\perp$

$$x_+ = x_0 + x_1 \qquad x_- = x_0 - x_1$$

$$\underbrace{x_0^2 - x_1^2}_{x_+x_-} - \mathbf{x}_\perp^2 = -R^2 \qquad x_- = \frac{\mathbf{x}_\perp^2 - R^2}{x_+}$$

$$\begin{aligned} ds^2 &= dx_+ dx_- - d\mathbf{x}_\perp^2 \\ dx_- &= \frac{2\mathbf{x}\cdot d\mathbf{x}}{x_+} - \frac{dx_+}{x_+^2}(\mathbf{x}_\perp^2 - R^2) \\ ds^2 &= dx_+ \left(\frac{2\mathbf{x}\cdot d\mathbf{x}}{x_+} - \frac{dx_+}{x_+^2}\left(\mathbf{x}^2 - R^2\right)\right) - d\mathbf{x}^2 \end{aligned}$$

  To remove the ugly cross-term, introduce $\mathbf{v} = f(x_+)\mathbf{x}$. So

$$\begin{aligned} d\mathbf{v} &= f' dx_+ \mathbf{x} + f d\mathbf{x} \\ d\mathbf{v}^2 &= (f')^2 \mathbf{x}^2 dx_+^2 + 2ff'\mathbf{x}\cdot d\mathbf{v} + f^2 d\mathbf{x}^2 \\ -d\mathbf{x}^2 &= \frac{1}{f^2}\left(-d\mathbf{v}^2 + (f')^2\mathbf{x}^2 dx_+^2 + 2ff'\mathbf{x}\cdot d\mathbf{x}\right) \end{aligned}$$

  The x-term cancels if $\frac{f'}{f} = -\frac{1}{x_+}$, $f = \pm\frac{1}{x_+}$ $(x_+ > 0)$.

  Thus with $\mathbf{v} \equiv \frac{\mathbf{x}}{x_+}$

$$\begin{aligned} ds^2 &= dx_+ \left(-\frac{dx_+}{x_+^2}(\mathbf{x}^2 - R^2)\right) - x_+^2 d\mathbf{v}^2 + \frac{1}{x_+^2}\mathbf{x}^2 dx_+^2 \\ &= R^2 \frac{dx_+^2}{x_+^2} - x_+^2 d\mathbf{v}^2 \end{aligned}$$

  Now with $x_+ \equiv Re^{t/R}$

$$ds^2 = dt^2 - R^2 e^{t/R} d\mathbf{v}^2$$

  which is an expanding flat spatial metric.

- Conformal coordinates

$$ds^2 = R^2 x_+^2 \left(\frac{dx_+^2}{x_+^4} - d\mathbf{v}^2\right)$$

  so with $x_+ \equiv \frac{1}{u}$

$$ds^2 = \frac{R^2}{u^2}\left(du^2 - d\mathbf{v}^2\right)$$

  de-Sitter space has the symmetry $SO(4,1)$ from the hyperboloid definition.

  In the light-front coordinates we have translation symmetries $\mathbf{v} \to \mathbf{v} + \mathbf{const}$. Where do these sit? See Appendix 3.

  We'll have much more to say about de-Sitter space later (inflation).

2.2 FRW (Friedman-Robertson-Walker) spacetimes

These are constructed by choosing one of the maximally symmetric paces and letting its overall scale vary with time. Thus

$$ds^2 = dt^2 - a^2(t)dl^2$$
$$dl^2 = \frac{du^2}{1+\kappa u^2} + u^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)$$

where

$$\begin{array}{lc} K=1 & \text{hyperbolic sections} \\ K=0 & \text{flat sections} \\ K=-1 & \text{spherical sections} \end{array}$$

These model spacetimes are homogeneous and isotropic, but evolving. They supply interesting first models for the observed universe averaged over large scales.

2.3 Curvature Calculations

Our master formulas (with correct signs) are

$$\omega_\mu^{ef} = \frac{e^{f\nu}}{2}\left(\partial_\mu e^e_\nu - \partial_\nu e^e_\mu + e_{a\mu}e^{e\rho}\partial_\rho e^a_\nu\right) - (e \leftrightarrow f)$$
$$R_{\mu\nu}{}^{\alpha\beta} = -F_{\mu\nu}{}^{ab}e^\alpha_a e^\beta_b$$
$$F_{\mu\nu}{}^{ab} = \partial_\mu\omega_\nu{}^{ab} - \partial_\nu\omega_\mu{}^{ab} - \omega_\mu{}^a{}_c\omega_\nu{}^c{}_b + \omega_\nu{}^a{}_c\omega_\mu{}^c{}_v$$

This is best exploited (for $g_{\mu\nu}$ diagonal) by using certain quasi-cartesian vierbeins

$$e^a_\alpha = \delta^a_\alpha g_a \qquad (\text{so } e^{a\alpha} = \eta^{a\alpha}g_a^{-1})$$

1st term: $$g_f^{-1}\frac{1}{2}\underbrace{\eta^{f\nu}\delta^e_\nu}_{\eta^{ef}}\partial_\mu g_e \to 0 \qquad (\text{symmetric in } e \leftrightarrow f)$$

2nd term: $$-\frac{1}{2}\eta^{f\nu}g_f^{-1}\delta^e_\mu\partial_\nu g_e = -\frac{1}{2}\eta^{f\nu}g_f^{-1}\delta^e_\mu\partial_\nu g_e$$

3rd term: $$\frac{1}{2}\underbrace{\eta^{f\nu}\eta_{a\mu}\eta^{e\rho}\eta^a{}_\nu}_{f,\nu,\mu,a \text{ all equal}}\cancel{g_f^{-1}}\cancel{g_a}g_e^{-1}\partial_\rho g_f = \frac{1}{2}\delta^f_\mu\eta^{e\rho}g_e^{-1}\partial_\rho g_f$$

So

$$\omega_\mu^{ef} = \delta^f_\mu\eta^{e\rho}g_e^{-1}\partial_\rho g_f - \delta^e_\mu\eta^{f\rho}g_f^{-1}\partial_\rho g_e$$

mnemonic: "μ matches on index, the other differentiates its g"

Example 1: 2d sphere (warm-up)

$$e^1_\theta = 1 = g_1 \qquad\qquad e^2_\phi = \sin(\theta) = g_2$$
$$\omega_\theta^{12} = 0 \qquad\qquad (\delta^e_\mu \text{ only, but } \partial_2 g_1 = 0)$$
$$\omega_\phi^{12} = +\cos(\theta) \qquad\qquad (\delta_2^{f=2}\eta^{1\rho}g_1^{-1}\partial_1 g_2)$$
$$F_{\theta\phi}{}^{12} = \partial_\theta\omega_\phi^{12} + (\text{vanishing})$$
$$= -\sin(\theta)$$

$$R_{\theta\phi}{}^{\theta\phi} = -(-\sin(\theta)\overbrace{e^\theta_1}^{1}\overbrace{e^\phi_2}^{\frac{1}{\sin(\theta)}}) = 1$$

By the way, this is the gauge field of a magnetic monopole (gauge group $SO(2) = U(1)$)!

Example 2: 3d sphere

$$
\begin{aligned}
e^1_\chi &= 1 & e^2_\theta &= \sin(\chi) & e^3_\phi &= \sin(\chi)\sin(\phi) \\
\omega^{12}_\chi &= 0 & \omega^{12}_\theta &= \cos(\chi) & \omega^{12}_\phi &= 0 \\
\omega^{13}_\chi &= 0 & \omega^{13}_\theta &= 0 & \omega^{13}_\phi &= \cos(\chi)\sin(\theta)
\end{aligned}
$$

$$
\begin{aligned}
F_{\chi\theta}{}^{12} &= \partial_\chi \omega^{12}_\theta = -\sin(\chi) \\
F_{\chi\phi}{}^{13} &= \partial_\chi \omega^{13}_\phi = -\sin(\chi)\sin(\theta) \\
F_{\theta\phi}{}^{13} &= \partial_\theta \omega^{13}_\phi - \omega^{12}_\theta \omega^{23}_\phi = \cos(\chi)\cos(\theta) - \cos(\chi)\cos(\theta) = 0 \\
F_{\theta\phi}{}^{23} &= \partial_\theta \omega^{23}_\phi - \omega^{21}_\theta \omega^{13}_\phi = -\sin(\theta) + \cos^2(\chi)\sin(\theta) = -\sin^2(\chi)\sin(\theta)
\end{aligned}
$$

Thus

$$-F_{\mu\nu}{}^{ab} = e^a_\mu e^b_\nu - e^b_\mu e^a_\nu$$

((?) The antisymmetry on indices μ, ν and a, b is automatic!)

or

$$
\begin{aligned}
R_{\mu\nu}{}^{ab} &= \delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu \\
R_\nu{}^\beta &= 2\delta^\beta_\nu \\
R &= 6
\end{aligned}
$$

Example 3: FRW cosmology (spatially flat case)
(Note: Mid-Latin indices are spatial, early Latin indices are internal)

$$
\begin{aligned}
e^0_t &= 1 & e^c_i &= \delta^c_i a(t) \\
ds^2 &= dt^2 - a(t)^2 d\mathbf{x}^2
\end{aligned}
$$

The only non-zero ω is

$$\omega^{0c}_i = \delta^c_i \dot{a}$$

The non-vanishing components of the field strength are

$$
\begin{aligned}
F_{0i}{}^{0c} &= \partial_0 \omega^{0c}_i = \delta^c_i \ddot{a} \\
F_{ij}{}^{cd} &= -\omega^{c0}_i \omega^{0d}_j + \omega^{c0}_j \omega^{0d}_i \\
&= \left(\delta^c_i \delta^d_j - \delta^c_k \delta^d_I\right) \dot{a}^2
\end{aligned}
$$

leading to the Ricci tensor components

$$
\begin{aligned}
R_0{}^0 &= -3\frac{\ddot{a}}{a} (= -F_{0i}{}^{0c} e^i_c) \\
R_i{}^l &= -F_{ij}{}^{cl} e^l_c e^j_d - F_{0i}{}^{0c} e^l_c \\
&= \left(-2\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right) \delta^l_i \\
R &= -6\frac{\dot{a}^2}{a^2} - 6\frac{\ddot{a}}{a}
\end{aligned}
$$

2.4 FRW Dynamics

The field equations (in $g^{\alpha\beta}$) are

$$R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R = 8\pi G \quad {}_{\nu}$$

We interpret $T^0{}_0 = \rho, T^i{}_j = -p\delta^i_j$ (Check 1: for electromagnetism, $T^{\mu}{}_{\mu} = 0, p = \frac{1}{3}\rho$)

From the preceding calculation

$$\begin{cases} 8\pi G\rho = 3\dfrac{\dot{a}^2}{a^2} \\ 8\pi G p = -2\dfrac{\ddot{a}}{a} - \dfrac{\dot{a}^2}{a^2} \end{cases} \tag{2}$$

$$\left(8\pi G(p+\rho) = 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right)\right)$$

Another important and appealing equation comes from differentiating the first of these and eliminating:

$$\begin{aligned} 8\pi G\dot{\rho} &= 6\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) \\ &= -3\cdot 8\pi G\frac{\dot{a}}{a}(\rho + p) \end{aligned}$$

or simply

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} \tag{3}$$

Another interesting thing is to see who's responsible for acceleration:

$$8\pi G(\rho + 3p) = -6\frac{\ddot{a}}{a}$$

There is a simple interpretation of (2) and (3):

(2): Imagine a test particle along for the ride. Gravity "outside" cancels (Birkhoff theorem). Conservation of particle's energy

$$\begin{aligned} \frac{m}{2}\overbrace{r^2\dot{a}^2}^{v^2} - \frac{G\cdot\frac{4\pi}{3}\rho r^3 a^3 m}{r} &= mkr^2 \\ \dot{a}^2 - \frac{8\pi G\rho}{3}a^2 &= k \end{aligned}$$

We have this with $k = 0$: neutral binding, critical "escape velocity"!

The non-zero values of k arise in FRW spaces with hyperbolic ($k > 0$) or (?) spherical ($k < 0$) spatial sections - see the problem set.

(3): Imagine work done by an expanding fluid against pressure; take it from mass-energy

$$\begin{aligned} \frac{d}{dt}\left(\frac{4\pi}{3}\rho a^3 r^3\right) &= -p\frac{d}{dt}\left(\frac{4\pi}{3}a^3 r^3\right) \\ \frac{d}{dt}\left(\rho a^3\right) &= -p\frac{d}{dt}(a^3) \\ \dot{\rho} &= -3(\rho + p)\frac{\dot{a}}{a} \end{aligned}$$

Appendix 3: Translations within $SO(4,1)$

Write the metric in block form:

$$-g = \left(\begin{array}{c|c} J & 0 \\ \hline 0 & 1 \end{array}\right) \qquad\qquad J \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The condition for a near-identity transformation

$$S = 1 + \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$$

to leave the metric invariant is

$$S^T g S \approx g; \quad \left(\begin{array}{c|c} a^T & c^T \\ \hline b^T & d^T \end{array}\right) \left(\begin{array}{c|c} J & 0 \\ \hline 0 & 1 \end{array}\right) + \left(\begin{array}{c|c} J & 0 \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \approx 0$$

or (to 1st order)

$$\begin{aligned} a^T J + Ja &= 0 \\ Jb - c^T &= 0 \\ b^T J - c &= 0 \\ d^T + d &= 0 \end{aligned}$$

With $a = d = 0$ the transformations $1 + \left(\begin{smallmatrix} 0 & b \\ b^T J & 0 \end{smallmatrix}\right)$ translate vectors $\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right)$ by $\left(\begin{smallmatrix} bs \\ b^T Jr \end{smallmatrix}\right)$, *i.e.* with things spelled out completely

$$b = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \eta & \phi \end{pmatrix}; \text{ so } b^T J = \begin{pmatrix} -\alpha & \delta \\ -\beta & \eta \\ -\gamma & \phi \end{pmatrix}$$

and

$$\Delta r = bs = \begin{pmatrix} \alpha s_1 + \beta s_2 + \gamma s_3 \\ \delta s_1 + \eta s_2 + \phi s_3 \end{pmatrix}$$

$$\Delta s = b^T Jr = \begin{pmatrix} -\alpha r_1 + \delta r_2 \\ -\beta r_1 + \eta r_2 \\ -\gamma r_1 + \phi r_2 \end{pmatrix}$$

Transformations with $\delta = -\alpha, \eta = -\beta, \phi = -\gamma$ leave $r_1 + r_2$ fixed while translating s through

$$\Delta s = -\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} (r_1 + r_2)$$

So $\mathbf{s}/(r_1 + r_2)$ is translated in the conventional way. In our previous notation this is

$$\frac{\mathbf{s}}{r_1 + r_2} = \frac{(x_2, x_3, x_4)}{x_0 + x_1} \qquad \frac{\mathbf{x}_\perp}{x_+} = \mathbf{v}$$

(This explains $\mathbf{v}$.)