Chapter 1

Galaxies: dynamics, potential theory, and equilibria

1.1 Dynamics of scattering

A natural way to begin is to ask the seemingly naive question "what is a galaxy?" The straightforward answer is that a galaxy is a system of stars and gas, like the Milky Way (the word *galaxy* comes from the Greek for "milk"). Prior to the last century, the Milky Way was not just the only known galaxy, but was thought to constitute the entire Universe. In 1926, Edwin Hubble showed that the Andromeda nebula was actually a separate galaxy, spanning 3° in the sky, another blow to our anthropocentric view of the Universe (if one was needed). The Earth is not the center of the Solar System, the Solar System is not the center of the Milky Way, and the Milky Way is just one of billions (and billions) of galaxies in the Universe.

Just as stars are held up by internal gas pressure, galaxies also have internal energy that prevents gravitational collapse, held in equilibrium by the virial theorem (e.g. Hansen and Kawaler 2004),

$$2T + U = 0, \tag{1.1}$$

where T is the internal kinetic energy of the particles (stars) and U is the negative gravitational potential energy. Like any self-gravitating system, the potential energy, up to constants of order unity, is given by

$$U \sim -\frac{GM^2}{R}, \tag{1.2}$$

where M and R are the mass and characteristic size of the system and G is Newton's gravitational constant[1]. The kinetic energy is given by

$$T = \frac{1}{2}Mv^2, \tag{1.3}$$

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[1]Although G is known to relatively poor accuracy in cgs units, in combination with the solar mass $GM_\odot$ is known quite well. We can specify G in solar mass units: $G = 4.30091851 \times 10^{-3} M_\odot^{-1}(km/s)^2 pc$. This is *very* useful given astronomers' habit of quoting masses in terms of $M_\odot$ and distances in terms of parsecs.
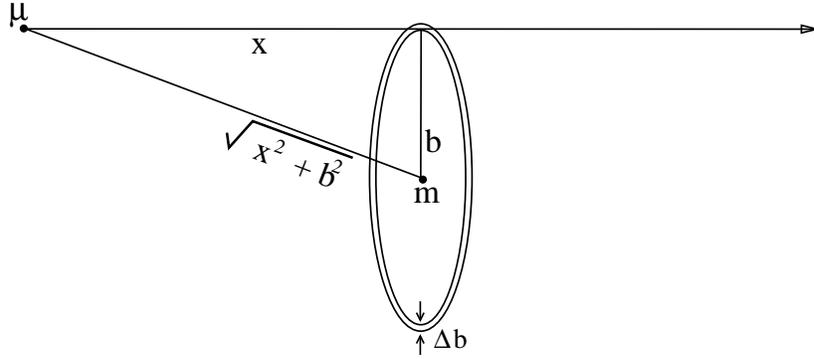
Figure 1.1: A star of mass μ travelling with velocity v passes a second star, with mass m at a distance of closest approach (impact parameter) b.

where v^2 is the average squared velocity for a particle in the system. The *gravitational potential* is then of order

$$\Phi \sim \frac{GM}{R} \sim v^2. \tag{1.4}$$

For a typical galaxy we have:

radius $R \sim 10\text{kpc}$

density $\rho \sim 0.1 M_\odot/\text{pc}^3$ and,

$N_{\text{stars}} \sim 10^{11}$.

The "thermal" velocity for individual stars within the galaxy is then $v \sim 210$ km/s.

Unlike the interiors of stars, where the mean free path (mfp) of particles is much shorter than the size of the star (mfp $\sim 1\mu m \ll R$), the mean free path of a star within a galaxy is much *larger* than the size of the galaxy, suggesting that the average star cannot be expected to scatter off another star during an orbit or even during its lifetime. How do we quantify scattering interactions? For a star with mass μ moving along a trajectory $x = vt$ with impact parameter b (the distance of closest approach if there were no deflection; see Fig. 1.1) relative to another star with mass m, the change in the component of velocity perpendicular to the direction of travel, $\Delta v_\perp$ is given by

$$\Delta v_\perp = \frac{F_\perp \Delta t}{\mu} = \frac{Gmb}{(x^2+b^2)^{3/2}}\Delta t. \tag{1.5}$$

Assuming a constant speed v during the interaction, this change in perpendicular velocity can be integrated over time, giving

$$\Delta v_\perp = \int_{-\infty}^{\infty} \frac{F_\perp}{\mu} dt = \frac{Gm}{b^2} \int_{-\infty}^{\infty} \frac{dt}{\left[1+\left(\frac{vt}{b}\right)^2\right]^{3/2}} = 2\frac{Gm}{bv}. \tag{1.6}$$

We call a scattering "strong" if $\Delta v_\perp = v$, so $b_{\text{strong}} = 2Gm/v^2$. The strong scattering cross section σ is therefore

$$\sigma = \pi b_{\text{strong}}^2 = \pi \left(\frac{2Gm}{v^2}\right)^2 \sim \frac{4\pi}{N^2}R^2, \tag{1.7}$$

where we have used the virial theorem to get $v^2 \sim GmN/R$. We now want to see how often a star might actually experience a strong encounter with another star. For low number densities n (an excellent assumption for galaxies), the probability of a strong scattering encounter during a single orbital crossing is given by

$$\begin{pmatrix} \text{probability of a} \\ \text{strong encounter} \\ \text{in one crossing} \end{pmatrix} \approx n\sigma R = \frac{N}{\frac{4}{3}\pi R^3} \cdot \frac{4\pi}{N^2}R^2 \cdot R = \frac{3}{N}, \tag{1.8}$$

a very small number for galaxies with $N \sim 10^{11}$! So we can safely say that strong scattering does not play a very important role in the orbits of stars within the galaxy.

We must nonetheless consider the cumulative effects of weak scattering from long-range gravitational interactions. Treating the encounters as a collection of steps in a random walk, we find

$$(\Delta v_\perp)_{\text{total}} = \sqrt{\sum_i (\Delta v_{\perp i})^2} \tag{1.9}$$

Integrating over all possible impact parameters, we find that over a time Δt, the accumulated perpendicular velocity is given by

$$\sum_i (\Delta v_{\perp i})^2 = \int_0^\infty \underbrace{2\pi v \Delta t\; b db}_{\text{volume element}}\; n \left(\frac{2Gm}{bv}\right)^2 = \Delta t \frac{8\pi Gm^2 n}{v} \int_0^\infty \frac{db}{b}. \tag{1.10}$$

In order to get a finite value for the integral, we must impose physical bounds on b. The upper limit is the size of the galaxy. The lower limit is the impact parameter at which a collision is judged to be strong. Taking $b_{\min} = Gm/v^2$ and $b_{\max} = R$, the integral can be evaluated as

$$\int_{b_{\min}}^{b_{\max}} \frac{db}{b} = \ln \Lambda = \ln \frac{Rv^2}{Gm} \sim \ln N \sim 25, \tag{1.11}$$

where we again use the virial theorem to substitute $R/Gm \sim N/v^2$.

For a system of stars in a galaxy (or globular cluster or any collisionless, gravitationally bound system), we can define a *relaxation time* for which the accumulated perpendicular velocity is comparable to the average velocity: $(\Delta v_\perp)_{\text{tot}} \sim v$. Combining equations (1.9, 1.10, and 1.11), we see that

$$t_{\text{relax}} \sim \frac{v^3}{8\pi G^2 nm^2} \frac{1}{\ln \Lambda}. \tag{1.12}$$

The number of orbits needed for the system to "relax" is given by

$$\frac{t_{\text{relax}}}{t_{\text{orbit}}} \sim \frac{v^3}{8\pi G^2 nm^2} \frac{1}{\ln \Lambda} \frac{v}{R} \sim \frac{v^4 R^2 \frac{4}{3}\pi}{8\pi G^2 Nm^2 \ln \Lambda} \sim \frac{N}{6\ln N}, \tag{1.13}$$

so it seems that weak scattering is *also* relatively insignificant for calculating orbits within galaxies. Evidently we must think of a galaxy not as a large collection of scatterers, but rather as a gravitational field that is created by and interacts with the matter in the galaxy.