**Last time**

$$\begin{array}{rcll} & & \underline{\text{label}} & \underline{\text{residual}} \\ P^- & = & p^- & + k^- \\ P_\perp & = & p_\perp & + k_\perp \end{array}$$

$\xi_{n,p}(x)$

$A^\mu_{n,p}(x)$

label operator $\mathcal{P}^\mu$

$$\mathcal{P}^\mu \ \xi_{n,p} = p^\mu \ \xi_{n,p}$$

$$\mathcal{P}^\mu \ \bar{\xi}_{n,p'} \Gamma \xi_{n,p} = (p^\mu - p'^\mu) \ \bar{\xi}_{n,p'} \Gamma \xi_{n,p}$$

$$i\partial^\mu \sum_{p\neq 0} e^{-ip\cdot x} \xi_{n,p}(x) = e^{-ix\cdot\mathcal{P}} \sum_{p\neq 0} (\mathcal{P}^\mu + i\partial^\mu) \ \xi_{n,p}(x)$$

labels conserved — often suppress this

residual momentum conserved

**Summary**

| Type | $(p^+, p^-, p^\perp)$ | Fields | Field Scaling |
|---|---|---|---|
| collinear | $(\lambda^2, 1, \lambda)$ | $\xi_{n,p}(x)$ | $\lambda$ |
| | | $(A^+_{n,p}, A^-_{n,p}, A^\perp_{n,p})$ | $(\lambda^2, 1, \lambda)$ |
| usoft | $(\lambda^2, \lambda^2, \lambda^2)$ | $q_{us}(x)$ | $\lambda^3$ |
| | | $A^\mu_{us}(x)$ | $\lambda^2$ |
| soft (later) | $(\lambda, \lambda, \lambda)$ | $q_{s,p}$ | $\lambda^{3/2}$ |
| | | $A^\mu_{s,p}$ | $\lambda$ |

## Collinear Lagrangian

Write $\psi = \xi_n + \xi_{\bar n}$ , $\xi_n = P_n \psi$ , $\xi_{\bar n} = P_{\bar n} \psi$

$P_n = \frac{\not n \not{\bar n}}{4}$ $\quad P_{\bar n} = \frac{\not{\bar n} \not n}{4}$

$$\mathcal{L} = \bar\psi \, i\not D \psi = (\bar\xi_{\bar n} + \bar\xi_n)\left( i\frac{\not n}{2}\bar n\cdot D + i\frac{\not{\bar n}}{2} n\cdot D + i\not D_\perp \right)(\xi_n + \xi_{\bar n})$$

$$= \bar\xi_n \frac{\not{\bar n}}{2} in\cdot D\, \xi_n + \underbrace{\bar\xi_{\bar n}\frac{\not n}{2} i\bar n\cdot D\, \xi_{\bar n}} + \bar\xi_n i\not D_\perp \xi_{\bar n} + \underbrace{\bar\xi_{\bar n} i\not D_\perp \xi_n}$$

So far we've done nothing, just written QCD in diff. vars.
Only $\xi_n$ components are big, so lets take only external $\xi_n$'s [do not couple current to $\xi_{\bar n}$ in path int.]

Integrate out $\xi_{\bar n}$

$\delta / \delta \bar\xi_{\bar n}$ : $\quad \frac{\not n}{2} i\bar n\cdot D\, \xi_{\bar n} + i\not D_\perp \xi_n = 0$

$$i\bar n\cdot D\, \xi_{\bar n} + \frac{\not{\bar n}}{2} i\not D_\perp \xi_n = 0$$

$$\xi_{\bar n} = \frac{1}{i\bar n\cdot D} i\not D_\perp \frac{\not{\bar n}}{2} \xi_n$$

Think of $\frac{1}{i\bar n\cdot\partial} f(x) = \int d^4p \, \frac{e^{-ip\cdot x}}{\bar n\cdot p} f(p)$ for inv. deriv.

Now

$$\mathcal{L} = \bar\xi_n \left( in\cdot D + i\not D_\perp \frac{1}{i\bar n\cdot D} i\not D_\perp \right)\frac{\not{\bar n}}{2} \xi_n$$

Next: introduce collinear & usoft gluon fields & phases $e^{-ip\cdot x}$

- recall $A_{us}^\mu$ has $p^2 \sim Q^2\lambda^4 \ll p_c^2 \sim Q^2\lambda^2$
  ie long wavelength, its like a classical background field as far as $A_n^\mu$ & $\xi_n$ are concerned

write $A^\mu = A_n^\mu + A_{us}^\mu$ [not quite right, but suffices here]

- Phase Redefinition $\quad i\partial^\mu \to \mathcal{P}^\mu + i\partial^\mu$
  get $e^{-ix\cdot\mathcal{P}}$ out front irrespective of number of fields we have ( $\frac{1}{i\bar{n}\cdot D}$ means we have Feyn rules with 0, 1, 2, 3, -- gluons)

$$\left.\begin{aligned} \xi_n &= \xi_{n,p} \\ in\cdot D &= \underset{\lambda^2}{in\cdot\partial} + \underset{\lambda^2}{g n\cdot A_{n,q}} + \underset{\lambda^2}{g n\cdot A_{us}} \end{aligned}\right\} \text{suppress } \sum_p, \sum_q$$

$$iD_\perp = \underbrace{(\mathcal{P}_\perp + gA^\perp_{n,q})}_{iD_\perp^c \sim \lambda} + \underbrace{(i\partial^\perp + gA^\perp_{us})}_{\lambda^2 \text{ drop it}}$$

$$i\bar{n}\cdot D = \underbrace{(\bar{\mathcal{P}} + g\bar{n}\cdot A_{n,q})}_{i\bar{n}\cdot D^c \sim \lambda^0} + \underbrace{(i\bar{n}\cdot\partial + g\bar{n}\cdot A_{us})}_{\lambda^2 \quad \text{drop it}}$$

Leading Order Action is $\mathcal{O}(\lambda^4)$ [ * $\lambda^{-4}$ from measure ]

$$\mathcal{L}^{(0)}_{qq} = e^{-ix\cdot\mathcal{P}}\, \bar{\xi}_{n,p'} \left[ in\cdot D + i\not{D}_\perp^c \frac{1}{i\bar{n}\cdot D_c} i\not{D}_\perp^c \right] \frac{\not{\bar{n}}}{2}\, \xi_{n,p}$$

- drop this if we remember to impose label conservation
- all fields are at $x$, derivatives $i\partial^\mu \sim \lambda^2$
  - action explicitly <u>local</u> at $Q\lambda^2$ scale
  - action local at $Q\lambda$ too ( $D_\perp$ in numerator, mom. space version of locality)
  - only non-local at $\sim Q$ scale
- terms are same size in power counting

<u>Repeat for Gluons</u>

$$\mathcal{L} = -\tfrac{1}{4}\, G^A_{\mu\nu} G^{\mu\nu A} = -\tfrac{1}{2}\, \mathrm{tr}\left[G_{\mu\nu} G^{\mu\nu}\right], \quad G^{\mu\nu} = \frac{i}{g}\left[D^\mu, D^\nu\right]$$

...

$$\mathcal{L}_{cg}^{(0)} = \frac{1}{2g^2}\, \mathrm{tr}\left\{\left(\left[i\hat{\mathcal{D}}^\mu + g A^\mu_{n,q},\ i\hat{\mathcal{D}}^\nu + g A^\nu_{n,q'}\right]\right)^2\right\} \quad + \text{ gauge fixing}$$

$$i\hat{\mathcal{D}}^\mu = \frac{i\bar{n}^\mu}{2}\, n\cdot D + \mathcal{P}_\perp^\mu + \frac{n^\mu}{2}\,\bar{\mathcal{P}}$$

see hep-ph/0109045

- terms dropped in constructing $\mathcal{L}_{\xi\xi}^{(0)}$, $\mathcal{L}_{cg}^{(0)}$ give $\mathcal{L}_{\xi\xi}^{(1)}$, $\mathcal{L}_{cg}^{(1)}$, ...

Argument so far was <u>tree level</u>. To go further we need symmetries (& power counting)

① Gauge Symmetry
② Reparameterization Invariance ] V. Useful
③ Spin Symmetry ?

③: Easiest in two-component form (rather than 4-components $\xi_n$ with $\frac{\not{n}\bar{\not{n}}}{4}\xi_n = \xi_n$)

$$\xi_n = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_n \\ \sigma^3\varphi_n\end{pmatrix}$$

$$\mathcal{L} = \varphi^\dagger_{n,p'}\left\{ i n\cdot D + iD_\perp^{c\,\mu}\frac{1}{i\bar{n}\cdot D_c}\, iD_\perp^{c\,\nu}\left(g^\perp_{\mu\nu} + i\epsilon^\perp_{\mu\nu}\sigma_3\right)\right\}\varphi_{n,p}$$

not $SU(2)$

just $U(1)$: helicity $h = \frac{i\epsilon_\perp^{\mu\nu}}{4}[\gamma_\mu,\gamma_\nu]$ generator

$h \sim \sigma_3$, spin along direction of motion

Broken by masses

Broken by non-pert effects

Useful in pert. theory

① Gauge Symmetry $\qquad U(x) = \exp\left[ i\alpha^A(x) T^A \right]$

Need to consider $U$'s which leave us within EFT

eg. $i\partial^\mu \alpha^A \sim Q\,\alpha^A$ then $\xi_n' = U(x)\,\xi_n$ would no longer have $p^2 \lesssim Q^2\lambda^2$

collinear $U(x)$ $\qquad i\partial^\mu U_c(x) \sim Q(\lambda^2, 1, \lambda)\, U_c(x) \leftrightarrow A^\mu_{n,q}$

usoft $U(x)$ $\qquad i\partial^\mu U_u(x) \sim Q(\lambda^2, \lambda^2, \lambda^2)\, U_u(x) \leftrightarrow A^\mu_{us}$

- two classes of gauge trnsfm for two gauge fields

- in momentum space we have convolutions for $U_c$

$$\xi_{n,p} \to \sum_q (U_c)_{p-q}\, \xi_{n,q}$$

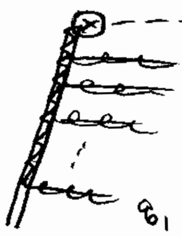
we'll write shorthand $\quad \xi_n \to U_c\, \xi_n$

Now $\quad q_{us} \xrightarrow{U_c} q_{us}$ $\quad$ since otherwise we give large mom. to an usoft field

<u>Aside</u> recall our heavy-to-light current

$\bar\xi_n \Gamma h_v^{us} \to \bar\xi_n U_c^\dagger \Gamma h_v^{us}$ $\quad$ is <u>not</u> gauge invariant

BUT we had to integrate out offshell propagators



$q_m$

$q_1$

+ perms of assigning $q_1, \ldots, q_m$ ie. crossed graphs

$$= \Gamma \sum_{m=0}^{\infty} \sum_{\text{perms}} \frac{(-g)^m\, \bar n\cdot \epsilon^{a_1}_{n,q_1} \cdots \bar n\cdot \epsilon^{a_m}_{n,q_m}}{\bar n\cdot q_1\;\bar n\cdot(q_1+q_2) \cdots \bar n\cdot\left(\sum q_i\right)}$$

$$* \quad T^{a_m} \cdots T^{a_1}$$

$$= \Gamma\, W$$

$\bar n\cdot A^{a_i}_{n,q_i} \to \bar n\cdot \epsilon^{a_i}_{n,q_i}$

we had first term previously $\quad -\frac{g\,\bar{n}^\mu}{\bar{n}\cdot q}\,\Gamma T^a$

Here $W$ is a <u>Wilson Line</u>

Short form $\quad W = \left[\sum_{\text{perms}} \exp\left(\frac{-g}{\bar{\mathcal{P}}}\,\bar{n}\cdot A_{n,q}(x)\right)\right]$

If we set residual coordinate $x=0$ then Fourier transform $\quad W = W(y,-\infty) = P\exp\left(ig\int_{-\infty}^{y} ds\; \bar{n}\cdot A(s\bar{n})\right)$

ie like $\quad \bar{\xi}_n(y)\, W(y,-\infty)\, h_v(-\infty)$

short dist. $y$ $\qquad$ usoft field at "long" dist. & doesn't see short dist. interactions

Now $W \to U_c W$ & $\quad \bar{\xi}_n W \Gamma h_v$ is <u>invariant</u>

<u>End Aside</u>

## Gauge Transformations

| | | $U_c$ | $U_{us}$ | $U_{global}$ |
|---|---|---|---|---|
| collinear | $\xi_{n,p}$ | $U_c\,\xi_{n,p}$ | $U_{us}\,\xi_{n,p}$ | easy |
| | $A_{n,p}$ | $U_c A_{n,p} U_c^\dagger + \frac{i}{g} U_c[i\hat{D}^\mu, U_c^\dagger]$ | $U_{us} A_{n,p} U_{us}^\dagger$ | --- |
| | $W$ | $U_c W$ | $U_{us} W U_{us}^\dagger$ | -- |
| usoft | $q_{us}$ | $q_{us}$ | $U_{us}\, q_{us}$ | -- |
| | $A_{us}$ | $A_{us}$ | $U_{us}\left(A_{us}^\mu + \frac{i\partial^\mu}{g}\right)U_{us}^\dagger$ | --- |
| | $Y$ | $Y$ | $U_{us}\, Y$ | -- |

- homogeneous in $\lambda$, recall $i\hat{D}^\mu$ has $in\cdot D$ in it
  $U_{us} A_{n,p} U_{us}^\dagger$ is like background field trasfm of quantum field $A_{n,p}$

- Gauge Symmetry ties together

$$i n\cdot D = i n\cdot\partial + g n\cdot A_n + g n\cdot A_{us}$$
$$i D_\perp^c$$
$$i\bar n\cdot D^c$$

Mass Dimension & p.c. means either $i n\cdot D \sim \lambda^2$

or $\frac{1}{\bar P}(i D_\perp)^2 \sim \lambda^2$ (no other $\lambda^2$ ops)

What about coeff. between $i n\cdot D$ & $i\cancel{D}_\perp \frac{1}{i\bar n\cdot D} i\cancel{D}_\perp$ ?

What about other operators like

$$\bar\xi_n \, i D_{\perp c}^\mu \frac{1}{i\bar n\cdot D} i D_{\perp,c}^\mu \frac{\bar{\cancel{n}}}{2} \xi_n \quad ?$$

(vi) <u>Reparameterization Invariance (RPI)</u>

$n, \bar n$ break Lorentz Inv. $n^\mu M_{\mu\nu}$, $\bar n^\mu M_{\mu\nu}$

(only $\epsilon_\perp^{\mu\nu} M_{\mu\nu}$ preserved)

rotations about 3-axis

3 types of RPI which keep $n^2 = \bar n^2 = 0$, $n\cdot\bar n = 2$

| | | |
|---|---|---|
| I $\quad n \to n + \Delta_\perp$ | II $\quad n \to n$ | III $\quad n \to e^\alpha n$ |
| $\quad\bar n \to \bar n$ | $\quad\bar n \to \bar n + \epsilon_\perp$ | $\quad\bar n \to e^{-\alpha}\bar n$ |

type III is simple! implies for any operator with an $n^\mu$ we have corresponding $n$ in denominator or a corresponding $\bar n$ in numerator

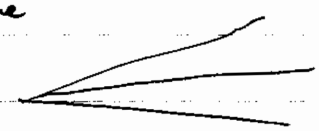
eg. $\mathcal{L}_{qq}^{(0)}$ had $\bar{\cancel{n}} \frac{1}{i\bar n\cdot D_c}$ ✓ , $\bar{\cancel{n}}\, n\cdot D$ ✓

can't have $\bar{\cancel{n}}\,\bar n\cdot D$

Power Counting $\left.\begin{array}{l}\Delta_\perp \sim \lambda \\ \epsilon_\perp \sim \lambda^0, \ \alpha \sim \lambda^0\end{array}\right\}$ max power that leaves scaling of collinear momenta intact

ie we only care about restoring Lorentz Inv. for the set of fluctuations described by SCET

stopped here ↓



Find
under I
$$n\cdot D \to n\cdot D + \Delta^\perp\cdot D^\perp$$
$$D^\perp_\mu \to D^\perp_\mu - \frac{\Delta^\perp_\mu}{2}\,\bar{n}\cdot D - \frac{\bar{n}_\mu}{2}\,\Delta^\perp\cdot D^\perp$$
$$\bar{n}\cdot D \to \bar{n}\cdot D$$
$$\xi_n \to \left(1 + \frac{\Delta\!\!\!/_\perp\, \bar{n}\!\!\!/}{4}\right)\xi_n$$
$$W \to W$$

under II
$$n\cdot D \to n\cdot D$$
$$D^\perp_\mu \to D^\perp_\mu - \frac{\epsilon^\perp_\mu}{2}\,n\cdot D - \frac{n_\mu}{2}\,\epsilon^\perp\cdot D^\perp$$
$$\bar{n}\cdot D \to \bar{n}\cdot D + \epsilon^\perp\cdot D^\perp$$
$$\xi_n \to \left(1 + \frac{\epsilon\!\!/_\perp}{2}\,\frac{1}{i\bar{n}\cdot D}\,i D\!\!\!\!/_\perp\right)\xi_n$$
$$W \to \left[\left(1 - \frac{1}{i\bar{n}\cdot D}\,i\epsilon^\perp\cdot D_\perp\right)W\right]$$

$$V^\mu = \frac{n\cdot V}{2}\,\bar{n}^\mu + \frac{\bar{n}\cdot V}{2}\,n^\mu + V_\perp^\mu$$ invariant under I, II, III