and again the ω_i's will be fixed by momenta that are external to collinear loops. An example where this would not be true is if we had the same collinear direction n in two or more of our building blocks, such as

$$\int d\omega_1\, d\omega_2\; C(\omega_1, \omega_2)\big[\overline{\chi}_{n,\omega_1} \bar{n}\!\!\!/\, \chi_{n,\omega_2}\big]\,. \tag{7.63}$$

For this operator one combination of ω_1 and ω_2 will be fixed by momentum conservation, while the other combination will involve collinear loop momenta. This will lead to anomalous dimension equations of a more complicated form, involving convolutions such as

$$\mu\frac{d}{d\mu}C(\mu,\omega) = \int d\omega'\gamma(\mu,\omega,\omega')C(\mu,\omega')\,. \tag{7.64}$$

Indeed, the operator in Eq. (7.64) is responsible for several classic evolution equations: i) DIS where we have DGLAP evolution for the parton distribution functions $f_{i/p}(\xi)$, ii) hard exclusive processes like $\gamma^*\pi^0 \to \pi^0$ where we have Brodsky-Lepage evolution for the light-cone meson distributions $\phi_\pi(x)$, and iii) the deeply virtual Compton scattering process $\gamma^* p \to \gamma p'$ where the evolution is a combination of both of theses. It is interesting that all of these processes are sensitive to different projections of the evolution of the single operator given in Eq. (7.64). We will carry out an example of an evolution equation with a convolution in the next section, where we consider DIS and the DGLAP equation.

8 Deep Inelastic Scattering

(**ROUGH)** DIS is a rich subject, so for the purpose of these notes we will treat only aspects related to factorization and the renormalization group evolution with SCET. In particular we will demonstrate the factorization of momentum by showing that the forward DIS scattering amplitude can be written as an integral over hard coefficients times parton distribution functions.

8.1 Factorization of Amplitude

The scattering process is depicted in the figure. The hard scale Q of the process is defined by the photon

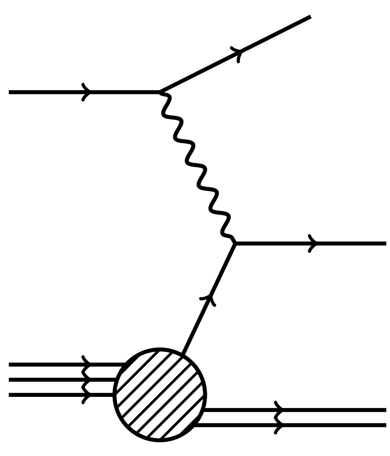

Figure 11: Deep Inelastic Scattering

momentum q^μ

$$q^2 = -Q^2 \tag{8.1}$$

and satisfies $Q^2 \gg \Lambda^2$. Our Bjorken variable x is defined in the standard way

$$x = \frac{Q^2}{2p \cdot q} \tag{8.2}$$

and with momentum conservation defined by $p^\mu + q^\mu = p_X^\mu$, we have

$$p_X^2 = \frac{Q^2}{x}(1-x) + m_p^2. \tag{8.3}$$

With this result we may determine the various energy regions of the process

Regions	Description
$\left(\frac{1}{x} - 1\right) \sim 1 \implies p_X^2 \sim Q^2$	Standard OPE Region
$\left(\frac{1}{x} - 1\right) \sim \frac{\Lambda}{Q} \implies p_X^2 \sim Q\Lambda$	Endpoint Region
$\left(\frac{1}{x} - 1\right) \sim \frac{\Lambda^2}{Q^2} \implies p_X^2 \sim \Lambda^2$	Resonance Region

Describe Parton Variables

We will consider our scattering process in the standard OPE region so that the final state has p_X^2 of order Q^2 and can consequently be integrated out. Conversely, the proton with its comparatively small invariant mass $p^2 \sim \Lambda^2$ may be treated as a collinear field. We analyze the process in the **Breit Frame** in which the perpendicular momentum component of q^μ is zero with

$$q^\mu = \frac{Q}{2}(\bar{n}^\mu - n^\mu). \tag{8.4}$$

The proton and final state momentum are then

$$\begin{aligned}
p^\mu &= \frac{n^\mu}{2}\bar{n} \cdot p + \frac{\bar{n}^\mu}{2} n \cdot p && (8.5)\\
&= \frac{n^\mu}{2}\bar{n} \cdot p + \frac{\bar{n}^\mu}{2}\frac{m_p^2}{\bar{n} \cdot p} && (8.6)\\
&= \frac{n^\mu}{2}\frac{Q}{x} + \cdots && (8.7)\\
p_X^\mu &= p^\mu + q^\mu && (8.8)\\
&= \frac{n^\mu}{Q} + \frac{\bar{n}^\mu}{2} Q \frac{(1-x)}{x}. && (8.9)
\end{aligned}$$

The cross section for DIS in terms of leptonic and hadronic tensors is

$$d\sigma = \frac{d^3k'}{2|\vec{k}'|(2\pi)^3}\frac{\pi e^4}{sQ^4} L^{\mu\nu}(k,k') W_{\mu\nu}(p,q) \tag{8.10}$$

where k and k' are the incoming and outgoing lepton momenta, respectively, and we have defined $q \equiv k' - k$, and $s \equiv (p+k)^2$. $L^{\mu\nu}(k,k')$ is the leptonic tensor computed using standard QFT methods and $W_{\mu\nu}(p,q)$ is the hadronic tensor which will occupy us in this section. $W_{\mu\nu}$ is related to the imaginary part of the DIS scattering amplitude by

$$W_{\mu\nu}(p,q) = \frac{1}{\pi}\mathrm{Im}T_{\mu\nu} \tag{8.11}$$

where

$$T_{\mu\nu}(p,q) = \frac{1}{2}\sum_{\text{spin}} \langle p| \hat{T}_{\mu\nu}(q) |p\rangle \qquad \hat{T}_{\mu\nu}(q) = i\int d^4x e^{iqx} T[J_\mu(x) J_\nu(0)]. \tag{8.12}$$

Taking J_μ to be an electromagnetic current, we may write

$$T_{\mu,\nu}(p,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) T_1(x,Q^2) + \left(p_\mu + \frac{q_\mu}{2x}\right)\left(p_\nu + \frac{q_\nu}{2x}\right) T_2(x,Q^2). \tag{8.13}$$

which satisfies current conservation, P, C, and T symmetries. Matching the $\hat{T}_{\mu\nu}(q)$ onto the most general leading order SCET operator for collinear fields in the n^μ direction and satisfying current conservation $q^\mu \hat{T}_{\mu\nu}$ we have

$$\hat{T}^{\mu\nu} \;\to\; \frac{g_\perp^{\mu\nu}}{Q}\left(O_1^{(i)} + \frac{O_1^g}{Q}\right) + \frac{(n^\mu+\bar{n}^\mu)(n^\nu+\bar{n}^\nu)}{Q}\left(O_2^{(i)} + \frac{O_2^g}{Q}\right) \tag{8.14}$$

where

$$O_j^{(i)} = \overline{\xi_{n,p}^i} W \frac{\bar{n}\!\!\!/}{2} C_j^{(i)}(\overline{\mathcal{P}}_+, \overline{\mathcal{P}}_-) W^\dagger \xi_{n,p}^i \tag{8.15}$$

$$O_j^g = \mathrm{Tr}[W^\dagger B_\perp^\lambda W\, C_j^g(\overline{\mathcal{P}}_+, \overline{\mathcal{P}}_-) W^\dagger B_{\perp\,\lambda} W] \tag{8.16}$$

$$\tag{8.17}$$

with $igB_\perp^\lambda$ and $\overline{\mathcal{P}}_\pm$ defined as

$$igB_\perp^\lambda \equiv [i\bar{n}\cdot D_n,\, iD_{n,\perp}^\lambda], \qquad \overline{\mathcal{P}}_\pm = \overline{\mathcal{P}}^\dagger \pm \overline{\mathcal{P}}. \tag{8.18}$$

The subscripts j in $O_j^{(i)}$ are arbitrary labels, similar to those found in (8.13), which differentiate the two parts of of $\hat{T}_{\mu\nu}$. The superscript (i) defines the flavor $(u, d, s, \text{etc.})$ of quarks and the superscript g in O_j^g stands for a gluon. In accord with their labels, $O_j^{(i)}$ will lead to the quark and anti-quark PDF and O_j^g will lead to the gluon PDF. The placement of factors of $\frac{1}{Q}$ is done in order to yield dimensionless Wilson coefficients. The fact that these Wilson coefficients are dimensionless can be understod by realizing that according to (8.12), $\hat{T}_{\mu\nu}$ has mass dimension 2.

In (8.14) there are both quark and gluon operators. However, with $\hat{T}_{\mu\nu}$ defined in terms of an electromagnetic current we can focus on the quarks and treat the gluons as an higher order contribution so that $\hat{T}_{\mu\nu}$ becomes

$$\hat{T}^{\mu\nu} \;\to\; \frac{g_\perp^{\mu\nu}}{Q} O_1^{(i)} + \frac{(n^\mu+\bar{n}^\mu)(n^\nu+\bar{n}^\nu)}{Q} O_2^{(i)}. \tag{8.19}$$

Returning to the quark operator $O_j^{(i)}$, we may introduce a convolution to separate the hard coefficients from the long distance operators

$$O_j^{(i)} = \int d\omega_1 d\omega_2\, C_j^{(i)}(\omega_+,\,\omega_-)[(\bar{\xi}_n W)_{\omega_1}\delta(\omega_1 - \overline{\mathcal{P}}^\dagger)\frac{\bar{n}\!\!\!/}{2}(W^\dagger \xi_n))_{\omega_2}\delta(\omega_2 - \mathcal{P})] \tag{8.20}$$

where $\omega_\pm = \omega_1 \pm \omega_2$. Our hope is to connect this operator to the PDF as a clear demonstration of factorization. The PDF for quarks is given by

$$f_{i/p}(\xi) = \int dy e^{-2i\xi\bar{n}\cdot py} \langle p|\,\bar{\xi}(y) W(y,-y) \bar{n}\!\!\!/ \xi(y)\,|p\rangle \tag{8.21}$$

and the PDF for anti-quarks is simply $\overline{f}_{i/p}(\xi) = -f_{i/p}(-\xi)$. In momentum space, we can write the matrix element in (8.21) as

$$\langle p|\,\bar{\xi}(y) W(y,-y) \bar{n}\!\!\!/ \xi(y)\,|p\rangle = \langle p|\,(\bar{\xi}_n W)_{\omega_1} \bar{n}\!\!\!/ (W^\dagger \xi_n))_{\omega_2}\,|p\rangle \tag{8.22}$$

$$= 4\bar{n}\cdot p \int_0^1 d\xi\, \delta(\omega_-) \tag{8.23}$$

$$\times\, [\delta(\omega_+ - 2\xi\bar{n}\cdot p) f_{i/p}(\xi) - \delta(\omega_+ + 2\bar{n}\cdot p)\overline{f}_{i/p}(\xi)]. \tag{8.24}$$

The delta function over ω_- sets $\omega_1 = \omega_2$. The other set of delta functions ensure that for $\omega_+ > 0$ we use quark PDF $f_{i/p}(z)$. and for $\omega_+ < 0$ we use anti-quark PDF $\overline{f}_{i/p}(z)$. Using these results we may rewrite our operator $O_j^{(i)}$ including spin averages as

$$\frac{1}{2}\sum_{\text{spin}} \langle p| O_j^{(i)} |p\rangle = \frac{1}{4}\int d\omega_1 d\omega_2\, C_j^{(i)}(\omega_+,\, \omega_-)[(\bar{\xi}_n W)_{\omega_1}\delta(\omega_1 - \overline{\mathcal{P}}^\dagger)\bar{\not{n}}(W^\dagger \xi_n))_{\omega_2}\delta(\omega_2 - \mathcal{P})] \tag{8.25}$$

$$= \frac{1}{4}\int d\omega_1 d\omega_2 C_j^{(i)}(\omega_+,\, \omega_-)4\bar{n}\cdot p \tag{8.26}$$

$$\times \int_0^1 d\xi\, \delta(\omega_-)[\delta(\omega_+ - 2\xi\bar{n}\cdot p)f_{i/p}(\xi) - \delta(\omega_+ + 2\bar{n}\cdot p)\overline{f}_{i/p}(\xi)] \tag{8.27}$$

$$= \bar{n}\cdot p\int_0^1 [C_j^i(2\bar{n}\cdot p\xi, 0)f_{i/p}(\xi) - C_j^i(-2\bar{n}\cdot p\xi, 0)\overline{f}_{i/p}(\xi)]. \tag{8.28}$$

Now, by charge conjugation invariance (reference), we have $C(-\omega_+,\, \omega_-) = -C(\omega_+,\, \omega_-)$ so that the final form of the spin averaged matrix element is

$$\frac{1}{2}\sum_{\text{spin}} \langle p| O_j^{(i)} |p\rangle = \bar{n}\cdot p\int_0^1 C_j^i(2\bar{n}\cdot p\xi, 0)[f_{i/p}(z\xi) + \overline{f}_{i/p}(z\xi)]. \tag{8.29}$$

We note that although we are using SCET$_{II}$ no soft gluons have appeared in our analysis. This fact can be understood by observing that our original operator

$$O_j^{(i)} = \overline{\xi_{n,p}^i} W \frac{\bar{\not{n}}}{2} C_j^{(i)}(\overline{\mathcal{P}}_+, \overline{\mathcal{P}}_-)W^\dagger \xi_{n,p}^i$$

is a color singlet and therefore decouples from any color-charge changing (i.e. gluon) interactions. With (8.29) we have the necessary result for a demonstration of factorization. Now all that is left to do is perform the matching of the full field theoretic operators $T_1(x, Q^2)$ and $T_2(x, Q^2)$ onto the operators $O_j^{(i)}$. Recalling our formula for $T_{\mu\nu}$ in terms of $\hat{T}_{\mu\nu}$, we have

$$T^{\mu\nu} = \frac{1}{2}\sum_{\text{spin}} \langle p| \hat{T}^{\mu\nu} |p\rangle \tag{8.30}$$

$$= \frac{g_\perp^{\mu\nu}}{Q}\frac{1}{2}\sum_{\text{spin}} \langle p| O_j^{(i)} |p\rangle + \frac{4n^\mu n^\nu}{Q}\frac{1}{2}\sum_{\text{spin}} \langle p| O_j^{(i)} |p\rangle\ . \tag{8.31}$$

This is the SCET amplitude. The QCD amplitude is

$$T_{\mu,\nu}^{SCET}(p,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)T_1(x,Q^2) + \left(p_\mu + \frac{q_\mu}{2x}\right)\left(p_\nu + \frac{q_\nu}{2x}\right)T_2(x,Q^2) \tag{8.32}$$

Writing this result in light-cone coordinates and using the Ward Identity ($q_\nu L^{\mu\nu} = q_\mu L^{\mu\nu} = 0$), and the fact that all terms proportional to $(\bar{n}_\mu - n_\mu) = \frac{2q_\mu}{Q}$ become zero upon contraction with $L_{\mu\nu}$, we have

$$T_{\mu\nu}^{QCD} = -g_{\mu\nu\perp}T_1(x,Q^2) + n_{\mu\nu}\left(\frac{Q^2}{4x^2}T_2(x,Q^2) - T_1(x,Q^2)\right) \tag{8.33}$$

We refer the reader to [?] for a full derivation of this result. Matching T^{QCD} onto T^{SCET}, yields the relations

$$-\frac{1}{2Q}\sum_{\text{spin}}\langle p|\,O_j^{(i)}\,|p\rangle = T_1(x,Q^2) \tag{8.34}$$

$$\frac{2}{Q}\sum_{\text{spin}}\langle p|\,O_j^{(i)}\,|p\rangle = \left(\frac{Q^2}{4x^2}T_2(x,Q^2) - T_1(x,Q^2)\right) \tag{8.35}$$

which, upon inversion, gives

$$T_1(x,Q^2) = -\frac{1}{2Q}\sum_{\text{spin}}\langle p|\,O_j^{(i)}\,|p\rangle \tag{8.36}$$

$$= -\frac{1}{x}\int_0^1 d\xi C_1^i(2\bar{n}\cdot p\xi,0)[f_{i/p}(\xi)+\overline{f}_{i/p}(\xi)] \tag{8.37}$$

$$T_2(x,Q^2) = \frac{8x^2}{Q^3}\sum_{\text{spin}}\langle p|\,O_j^{(i)}\,|p\rangle - \frac{2x^2}{Q^3}\sum_{\text{spin}}\langle p|\,O_j^{(i)}\,|p\rangle \tag{8.38}$$

$$= \frac{4x}{Q^2}\int_0^1 d\xi\left(4C_2^{(i)}(2\bar{n}\cdot p\xi,0) - C_1^{(i)}(2\bar{n}\cdot p\xi,0)\right)[f_{i/p}(\xi)+\overline{f}_{i/p}(\xi)]. \tag{8.39}$$

$$\tag{8.40}$$

where in the Breit Frame $x = \frac{Q^2}{2p\cdot q} = \frac{Q^2}{\bar{n}\cdot pn\cdot q} = \frac{Q}{\bar{n}\cdot p}$. With the definition

$$H_j(z) \equiv C_j(2Qz,0,Q,\mu), \tag{8.41}$$

where the hard scale Q and the μ dependence has been made explicit, we have the final result

$$T_1(x,Q^2) = -\frac{1}{x}\int_0^1 d\xi\, H_1^{(i)}\left(\frac{\xi}{x}\right)[f_{i/p}(\xi)+\overline{f}_{i/p}(\xi)] \tag{8.42}$$

$$T_2(x,Q^2) = \frac{4x}{Q^2}\int_0^1 d\xi\left[4H_2^{(i)}\left(\frac{\xi}{x}\right) - H_1^{(i)}\left(\frac{\xi}{x}\right)\right][f_{i/p}(\xi)+\overline{f}_{i/p}(\xi)] \tag{8.43}$$

where the sum over i is implicit.

Remarks

- This result represents the general (to all orders in α_s) factorization for DIS. As promised we have the computable hard coefficients H_i weighted by the universal non-perturbative PDFs $f_{i/p}$ and $\overline{f}_{i/p}$.
- The coefficients C_j are dimensionless and can therefore only have $\alpha_s(\mu)\ln(\mu/Q)$ dependence on Q. This result is in accord with Bjorken Scaling.
- The μ in $H_i(\mu)$ and $f_{i/p}(\mu)$ is typically called the factorization scale $\mu = \mu_F$. There is also the renormalization scale as in $\alpha_s(\mu_R)$. In SCET μ is both the renormalization and factorization scale, since the same parameter μ is responsible for the running of the EFT coupling $\alpha_s(\mu)$ and for the EFT coupling $C_j(\mu)$.
- When we consider the tree level matching onto the wilson coefficients we find that $C_2 = 0$ implying the Callan-Gross relation

$$\frac{W_1}{W_2} = \frac{Q^2}{4x^2} \tag{8.44}$$

and that

$$C_1(\omega_+) = 2e^2Q_i^2\left[\frac{Q}{(\omega_+ - 2Q) + i\epsilon} - \frac{Q}{(-\omega_+ - 2Q) + i\epsilon}\right] \tag{8.45}$$

$$H_1 = -e^2Q_i^2\delta\left(\frac{\xi}{x} - 1\right) \tag{8.46}$$

8.2 Renormalization of PDF

(ROUGH) In this section we calculate the anomalous dimension of the parton distribution function. We define the PDF as

$$f_q(\xi) = \langle p_n|\,\overline{\chi}_n(0)\frac{\bar{\not{n}}}{2}\chi_{n,\omega}(0)\,|p_n\rangle \tag{8.47}$$

where $\omega = \xi\,\bar{n}\cdot p_n > 0$. Since we have a forward matrix element there is no need to consider a momentum label ω' on $\overline{\chi}_n$, by momentum conservation it would be fixed to $\omega' = \omega$. We renormalize our PDF in our EFT framework with dimensional regularization, noting that there are only collinear fields an no ultrasoft interactions for this example. Collinear loop processes can change ω (or ξ) and also the type of parton. The renormalized PDF operators are given in terms of bare operators as

$$f_i^{\text{bare}}(\xi) = \int d\xi'\, Z_{ij}(\xi,\xi')f_j(\xi',\mu). \tag{8.48}$$

The μ independence of the bare operators $f_i^{bare}(\xi)$ yields an RGE for the renormalized operators in $\overline{\text{MS}}$,

$$\mu\frac{d}{d\mu}f_i(\xi,\mu) = \int d\xi'\gamma_{ij}(\xi,\xi')f_j(\xi',\mu) \tag{8.49}$$

where

$$\gamma_{ij} = -\int d\xi'' Z_{ii'}^{-1}(\xi,\xi'')\mu\frac{d}{d\mu}Z_{i'j}(\xi'',\xi')\,. \tag{8.50}$$

At 1-loop we can take $Z_{ii}^{-1}(\xi,\xi'') = \delta_{ii}\,\delta(\xi-\xi'') + \cdots$ so that

$$\gamma_{ij}^{\text{1-loop}} = -\mu\frac{d}{d\mu}\big[Z_{ij}(\xi,\xi')\big]^{\text{1-loop}} \tag{8.51}$$

Computing the PDF at tree level, we obtain

$$= \underbrace{\overline{u}_n\frac{\not{n}}{2}u_n}_{p^-}\delta(\omega - p^-) = \delta(1-\omega/p^-) \tag{8.52}$$

At the 1-loop level there are multiple contributions the first contribution yields the computation

$$= -ig^2C_F\int d^dl\frac{p^-(d-2)l_\perp^2}{[l^2+i0]^2[(l-p)^2+i0]}\delta(l^- - \omega)\frac{\mu^{2\epsilon}e^{\epsilon\gamma_E}}{(4\pi)^\epsilon} \tag{8.53}$$

$$= \frac{2g^2}{(4\pi)^2}(1-\epsilon)^2\Gamma(\epsilon)e^{\epsilon\gamma_E}(1-z)\theta(z)\theta(1-z)\left(\frac{A}{\mu^2}\right)^{-\epsilon} \tag{8.54}$$

$$= \frac{\alpha_sC_F}{\pi}(1-z)\theta(z)\theta(1-z)\left[\frac{1}{2\epsilon} - 1 - \frac{1}{2}\ln\left(A/\mu^2\right)\right] \tag{8.55}$$

where $A = -p^+p^-z(1-z)$ with $z = \omega/p^-$ The next contribution is given by

$$=2ig^2C_F \int \frac{d^dl\,\overline{u}_n \frac{\not{\bar{n}}}{2}\frac{\not{n}}{2}\bar{n}\cdot l\not{\bar{n}} u_n}{(l^- - p^-)l^2(l-p)^2}[\overbrace{\delta(l^- - \omega)}^{\text{real}} - \overbrace{\delta(p^- - \omega)}^{\text{virtual}}] \tag{8.56}$$

$$=\frac{C_F\alpha_s(\mu)}{\pi}e^{\epsilon\gamma_E}\Gamma(\epsilon)\left[\frac{z\theta(z)\theta(1-z)}{(1-z)^{1+\epsilon}}\left(\frac{-p^-p^+z-i0}{\mu^2}\right)^{-\epsilon}\right. \tag{8.57}$$

$$\left. -\delta(1-z)\left(\frac{-p^-p^+z-i0}{\mu^2}\right)^{-\epsilon}\frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma 2-2\epsilon}\right] \tag{8.58}$$

We can simplify this result with use of the distribtuion identity.

$$\frac{\theta(1-z)}{(1-z)^{1+\epsilon}} = -\frac{\delta(1-z)}{\epsilon} + \mathcal{L}_0(1-z) - \epsilon\mathcal{L}_1(1-z) + \cdots \tag{8.59}$$

where the plus function $\mathcal{L}_n(x)$ is defined as

$$\mathcal{L}_n(x) = \left[\frac{\theta(x)\ln^n(x)}{x}\right] \tag{8.60}$$

and satisfies the following identities

$$\int_0^1 dx\,\mathcal{L}_n(x) = 0, \quad \int_0^1 \mathcal{L}_n(x)g(x) = \int_0^1 dx\,\frac{\ln^n x}{x}[g(x)-g(0)]. \tag{8.61}$$

With this replacement we find that the $1/\epsilon^2$ terms in the real and virtual terms cancel and the remaining $1/\epsilon$ is UV divergent. In the end the explicit contribution of this process is

$$=\frac{C_F\alpha_s(\mu)}{\pi}\left[\{\delta(1-z) + z\theta(z)\mathcal{L}_0(1-z)\}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^+p^-z-i0}\right)\right. \tag{8.62}$$

$$\left. -z\mathcal{L}_2(1-z)\theta(z) + \delta(1-z)\left(2-\frac{\pi^2}{6}\right)\right]. \tag{8.63}$$

The last conrtibution to the renormalized PDF is the wavefunction renormalization of the external fermions.

$$Fig() = \delta(1-z)(Z_\psi - 1) = \frac{\alpha_s C_F}{\pi}\left[-\frac{1}{4\epsilon} - \frac{1}{4} - \frac{1}{4}\ln\left(\frac{\mu^2}{-p^+p^- - i0}\right)\right]\delta(1-z) \tag{8.64}$$

There are additional contributions from diagrams such as those in (), but we will ignore these by assuming that the operator is not a flavor singlet. Summing the various contributions, we have

$$\text{Sum} = \frac{C_F\alpha_s(\mu)}{\pi}\left[\left\{\frac{3}{4}\delta(1-z) + z\theta(z)\mathcal{L}_0(1-z) + \right.\right.$$

$$\left.\left. + \frac{(1-z)}{2}\theta(z)\theta(1-z)\right\}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^+p^-z_i0}\right) + \text{finite function of } z\right]$$

$$=\frac{C_F\alpha_s(\mu)}{\pi}\left[\underbrace{\frac{1}{2}\left(\frac{1+z^2}{1-z}\right)_+}_{\text{Determines } Z_{qq}^{\text{1-loop}}}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{-p^+p^-z_i0}\right) + \cdots \text{finite function of z}\right] \tag{8.65}$$

If we let the total momentum of the hadronic state be $\hat{p}^-$. Then define $p^-/\hat{p}^- = \xi^-$. So that

$$z = \frac{\omega}{p^-} = \frac{\xi \hat{p}^-}{\xi' \hat{p}^-} = \frac{\xi}{\xi'} \tag{8.66}$$

Then our $Z_{qq}^{\text{1-loop}}$ becomes

$$Z_{qq}^{\text{1-loop}} = \delta(1-z) + \frac{1}{\epsilon}\frac{\alpha_s(\mu)}{2\pi} C_F \theta z \theta(1-z) \Big(\frac{1+z^2}{1-z}\Big)_+ . \tag{8.67}$$

And usng

$$\gamma_{ij} = -\mu \frac{d}{d\mu} \mathcal{Z}_{ij}(z,\mu), \quad \mu \frac{d}{d\mu}\alpha_s(\mu) = -2\epsilon \alpha_s(\mu_) + \beta[\alpha_s(\mu)] \tag{8.68}$$

we then obtain the our final result

$$\gamma_{qq}(\xi,\xi') = \frac{C_F \alpha_s \mu}{\pi} \frac{\theta(\xi'-\xi)\theta(1-\xi')}{\xi'} \left(\frac{1+z^2}{1-z}\right)_+ \tag{8.69}$$

which is the Aliterelli - Parisi (DGLAP) quark anomalous dimension at one-loop.

8.3 General Discussion on Appearance of Convolutions in $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$

9 Dijet Production, $e^+e^- \to 2$ jets

(ROUGH)

The production of jets at an e^+e^- collider has historically been very important. Measurements of various jet in e^+e^- collisions were used to validate QCD as the correct theory of the strong interaction, and to this day, even 10 years after the LEP has been turned off, measurements of event shape distributions are being used to study the nature of the strong interaction and to determine fundamental constants of nature such as the coupling constant of the strong interaction.

The dominant kinematical situation in $e^+e^- \to$ jets is to produce two jets, but of course a larger number of jets can be obtained by the emission of additional hard strongly interacting particles. In this section we will discuss the production of two jets in e^+e^- collisions, which is to say the production of energetic particles in two back-to-back directions, accompanied only by usoft radiation in arbitrary regions of phase space.

Clearly, the question whether we have 2 or more jets has to be determined on an event by event basis, and there are many possible observables which can distinguish 2-jet events from events with more than 2 jets. The most natural definition might be to use a jet finding algorithm, and select those events with exactly two hard jets as defined by this algorithm. However, there is another set of observables which can be used to identify 2-jet events, and which are much easier to analyze theoretically. This class of observables are called event shapes, with the most well known event shape variable being thrust. In this section, we will only discuss the thrust distribution in e^+e^- collisions, but it should be clear from the discussion how one can extend the results to other event shape variables or other 2-jet observables.

9.1 Kinematics, Expansions, and Regions

The thrust of an event is defined as follows:

$$T = \max_{\vec{n}_T} \frac{\sum_i |\vec{p}_i \cdot \vec{n}_T|}{\sum_i |\vec{p}_i|} \tag{9.1}$$

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