Chapter 3: Duality Toolbox

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Lecture 22

Note: parallel transport of such an "external quark" gives a slightly different object from that in an ordinary gauge theory, as it couples also to the scalar fields in $\mathcal{N}=4$ SYM. Geometrically, strings pull D-branes. One can show

$$W(C) = Tr\mathcal{P}\exp i\int ds(A_\mu \frac{dx^\mu}{ds} + \vec{n}\cdot\vec{\Phi}\sqrt{\dot{x}^2}) \tag{1}$$

where $\vec{n}$ is a unit vector on S^5 and $\vec{\Phi}$ is six scalar fields of $\mathcal{N}=4$ SYM.

Now consider this "quark" traverses some loop C on the boundary. Since (i) the quark is the end point of a string in AdS, C must be the boundary of a string worldsheet Σ, *i.e.* $C=\partial\Sigma$; (ii) the "partition function" for this quark system is $\langle W(C)\rangle$, we thus expect

$$\langle W(C)\rangle = Z_{string}[\partial\Sigma = C] \tag{2}$$

which is the single string partition function whose worldsheet has boundary C. We know

$$Z_{string}[\partial\Sigma=C] = \int_{\partial\Sigma=C} DXe^{iS_{string}} \tag{3}$$

where $S_{string} = S_{NG}$ or $S_{Polyakov}$, *i.e.*

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det h_{\alpha\beta}} \qquad h_{\alpha\beta} = g_{MN}\partial_\alpha X^M\partial_\beta X^N \tag{4}$$

Recall the Maldacena limit

$$\begin{aligned} g_s \to 0: &\ \text{neglect other topologies (no splitting and joining of strings)} \\ \alpha' \to 0: &\ \text{can evaluate path integral by saddle-point approximation (no fluctuation)} \end{aligned}$$

and this limit is equivalent to $N\to\infty$ and $\lambda\to\infty$. Under this limit, we should expect

$$\langle W(C)\rangle = Z_{string}[\partial\Sigma=C] = e^{iS_{cl}[\partial\Sigma=C]} \tag{5}$$

where S_{cl} is the action evaluated at a classical string solution.

Let us see some examples. The simplest one is a static quark, which connects a single string stretching to the interior of AdS. We know such an isolated Wilson loop evaluated as $\langle W(C)\rangle = e^{-iMT}$ where M is the mass of the quark. On the bulk side, in Poincare patch,

$$ds^2 = \frac{r^2}{R^2}(-dt^2+d\vec{x}^2) + \frac{R^2}{r^2}dr^2 \qquad (r=\frac{R^2}{z}) \tag{6}$$

using reparametrization freedom on the worldsheet, one can choose the coordinate on worldsheet as $\sigma^\alpha \equiv (\tau,\sigma) = (t,r)$. One obvious solution is

$$X^i(\sigma,\tau) = const \qquad (static\ string) \tag{7}$$

where the worldsheet metric becomes

$$ds^2_{ws} = g_{MN}\partial_\alpha X^M\partial_\beta X^N d\sigma^\alpha d\sigma^\beta = -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}dr^2 \tag{8}$$

Plug into the Nambu-Goto action, we get

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det h} = -\frac{1}{2\pi\alpha'}\int dt\int_0^\infty dr = -\frac{1}{2\pi\alpha'}T\Lambda \tag{9}$$

where Λ is the cutoff of r. This shows the mass of the quark should be $M = \frac{\Lambda}{2\pi\alpha'}$ as expected from D-brane calculation. The infinite mass refers to "external quark" by design. If we write the result in terms of z, introducing $\epsilon = R^2/\Lambda$ as the short-distance cutoff, one get

$$M = \frac{1}{2\pi\alpha'}\frac{R^2}{\epsilon} = \frac{\sqrt{\lambda}}{2\pi\epsilon} \qquad (\sqrt{\lambda} = \frac{R^2}{\alpha'}) \tag{10}$$

This corresponds to self energy in strong coupling of CFT on the boundary. Recall in QED Wilson loop calculation, $E_{self} \sim e^2/\epsilon \sim \alpha/\epsilon$, the dependence on the coupling constant is proportional to α not like here where it goes like $\alpha^{1/2}$.

The second example is the static potential between a quark and anti-quark. The Wilson loop is as follows

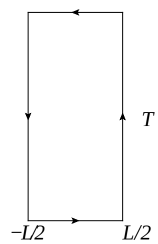

Figure 1: Square Wilson loop

In this picture $T \gg L$. As explained before, this corresponds to a static pair of quark and anti-quark with distance L. The total energy is $E_{tot} = 2M + V(L)$ and Wilson loop is evaluated as $\langle W(C)\rangle = e^{-iE_{tot}T}$. We will see how gravity will help us to calculate the potential $V(L)$ for $N \to \infty$ and $\lambda \to \infty$.

Choose $\sigma^\alpha \equiv (\tau, \sigma) = (t, z)$ for string coordinate. We would have the string hanging from two quarks on the boundary in AdS as shown below.

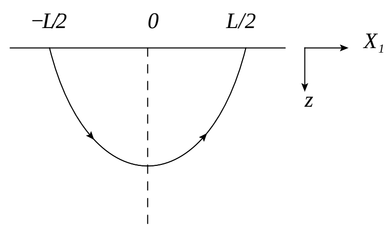

Figure 2: String hanging in AdS

Since T is very large, translation symmetry in time requires that $X_1 = X_1(\sigma)$ and $X^i = const$. Alternatively, we can also choose the worldsheet parameter as $\sigma^\alpha \equiv (\tau, \sigma) = (t, X_1)$ and $z = z(\sigma)$ is the position of the string with boundary condition $z(\pm L/2) = 0$. Then the worldsheet metric becomes

$$ds^2_{ws} = \frac{R^2}{z^2}(-d\tau^2 + (1 + z'^2)d\sigma^2) \tag{11}$$

where $z' \equiv dz/d\sigma$ which implies the action to be

$$S_{NG} = -\frac{R^2}{2\pi\alpha'}T\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{d\sigma}{z^2}\sqrt{1+z'^2} = -\frac{R^2}{\pi\alpha'}T\int_0^{\frac{L}{2}}\frac{d\sigma}{z^2}\sqrt{1+z'^2} \tag{12}$$

where in the last step we used the reflection symmetry $z(\sigma) = z(-\sigma)$. Now we need to extremize it to find $z(\sigma)$. One expect the integral to be divergent near $z = 0$ as S_{NG} contains contribution $2MT$ but we can cut off at $z = \epsilon$. With the self energy obtain above, we get the potential to be

$$V(L) = \frac{\sqrt{\lambda}}{\pi}\int_\epsilon^{\frac{L}{2}}\frac{d\sigma}{z^2}\sqrt{1+z'^2} - 2\frac{\sqrt{\lambda}}{2\pi\epsilon} = \frac{\sqrt{\lambda}}{\pi}\left(\int_\epsilon^{\frac{L}{2}}\frac{d\sigma}{z^2}\sqrt{1+z'^2} - \frac{1}{\epsilon}\right) \tag{13}$$

We define $\mathcal{L}$ as $\frac{1}{z^2}\sqrt{1+z'^2}$. Since $\mathcal{L}$ does not depend on σ explicitly, the Hamiltonian (canonical momentum with respect to σ) is a constant,

$$z'\Pi_z - \mathcal{L} = const \qquad \Pi_z = \frac{\partial \mathcal{L}}{\partial z'} \tag{14}$$

which can be solved out to be

$$\frac{1}{z^2\sqrt{1+z'^2}} = const \tag{15}$$

From reflection symmetry, at $\sigma = 0$, $z'(0) = 0$ and $z(0) = z_0$, we know the constant is exactly $1/z_0^2$. Hence we get

$$z'^2 = \frac{z_0^4 - z^4}{z^4} \tag{16}$$

which can be easily integrated to get z. Note z_0 can be fixed by requiring $z(L/2) = 0$ as

$$z_0 = L\frac{\sqrt{\pi}}{2}\frac{\Gamma(1/4)}{\Gamma(3/4)} \tag{17}$$

Plug (16) into (13), we have

$$V(L) = \frac{\sqrt{\lambda}}{\pi}\left(z_0^2\int_\epsilon^{z_0}\frac{dz}{z^2\sqrt{z_0^4 - z^4}} - \frac{1}{\epsilon}\right) = \frac{\sqrt{\lambda}}{\pi z_0}\left(\int_{\frac{\epsilon}{z_0}}^1 \frac{dy}{y\sqrt{1-y^2}} - \int_{\frac{\epsilon}{z_0}}^\infty \frac{dy}{y^2}\right) = -\frac{\sqrt{\lambda}}{L}\frac{4\pi^2}{\Gamma^4(1/4)} \tag{18}$$

Remarks

1. This potential is finite and negative, which means the interaction is attractive.
2. L^{-1} dependence same as coulomb potential is from scale invariance.
3. $\sqrt{\lambda}$ dependence on coupling constant is the result of strong coupling predicted from gravity. In weak coupling case, $V \propto -\frac{\lambda}{L}$.
4. $z_0 \propto L$ shows the IR/UV connection since larger z_0 corresponds to lower energy binding of the quark pair on the boundary.

3.2: GENERALIZATIONS

3.2.1: FINITE TEMPERATURE

So far we have following duality

$$\begin{aligned} \text{string in AdS}_5 \times S^5 &\iff \mathcal{N} = 4 \text{ SYM} \\ \text{normalizable solution} &\iff \text{state} \\ \text{pure AdS}_5 \times S^5 &\iff \text{vacuum} \end{aligned}$$

A natural question raises: what does the thermal state in SYM correspond to? The gravity description should satisfy:

1. It is asymptotic AdS_5 (normalizable)
2. It has a finite temperature T and satisfies all laws of thermodynamics
3. For Poincare patch, translationally invariant and rotationally invariant along boundary directions.

Regarding these conditions, here are two candidates:

1. Thermal gas in AdS
2. Black hole.

The thermal gas lives in AdS can be described by Euclidean AdS metric

$$ds^2 = \frac{R^2}{z^2}(d\tau^2 + dz^2 + d\vec{x}^2) \tag{19}$$

with periodicity $\tau \sim \tau + \beta$. Furthermore, for fermions we should require the partition function to be anti-periodic in τ. But this solution has two disadvantages, the first is that there is a curvature singularity at $z \to \infty$; the second is that strings winding around τ direction develop techyons, which will be unstable.

For black hole, we need to find a solution with an event horizon which is topologically $\mathbb{R}^{d-1}$. Taking the ansatz

$$ds^2 = \frac{R^2}{z^2}(-f(z)dt^2 + d\vec{x}^2) + \frac{R^2}{z^2}g(z)dz^2 \tag{20}$$

we can solve Einstein equations to get

$$f(z) = 1/g(z) = 1 - \frac{z^d}{z_0^d} \tag{21}$$

where z_0 is a constant, which characterize the position of horizon. Using the standard trick going to Euclidean signature, one finds

$$\beta = \frac{1}{T} = \frac{4\pi}{d}z_0 \implies T = \frac{d}{4\pi z_0} \tag{22}$$

This is the temperature measured in boundary, and $z_0 \propto T^{-1}$ shows again the IR/UV connection. Now we can obtain thermodynamical behavior of strongly coupled $\mathcal{N} = 4$ SYM ($N \to \infty$ and $\lambda \to \infty$) from black hole thermodynamics ($d = 4$). The entropy of black hole is

$$S_{BH} = \frac{A_3}{4G_5} \qquad A_3 = \frac{R^3}{z_0^3}\int dx_1 dx_2 dx_3 \tag{23}$$

We can define the entropy density as

$$s = S/\int dx_1 dx_2 dx_3 = \frac{R^3}{z_0^3}\frac{1}{4G_5} = \frac{\pi^2}{2}N^2T^3 \qquad (\frac{G_5}{R^3} = \frac{\pi}{2N^2}) \tag{24}$$

which is proportional to N^2 as expected from CFT entropy. One can also obtain the energy density and pressure from $\langle T_{\mu\nu}\rangle$, which can be calculated from the counterpart of $\mathcal{O}$ in scalar story, *i.e.*

$$\langle T_{\mu\nu}\rangle \propto \frac{1}{z_0^4} \sim T^4 \tag{25}$$

as expected for a CFT in $d = 4$. But getting the precise numerical factors takes some efforts. It is easier to use thermodynamics:

$$s = -\frac{\partial f}{\partial T} \implies f = -\frac{\pi^2}{8}N^2T^4 \tag{26}$$

$$\implies e = f + Ts = \frac{3\pi^2}{8}N^2T^4 \tag{27}$$

where f is free energy density and e is energy density. Note these classical gravity results are only valid at $\lambda \to \infty$.

Now we can compare with free theory results:

$$s_{\lambda=0} = (8 + 8 \times \frac{7}{8}) \times \frac{2\pi^2}{45}T^3(N^2 - 1) = \frac{2}{3}\pi^2N^2T^3 \qquad (N \to 0) \tag{28}$$

where the first 8 is for 8 bosons and the second is for 8 fermions in $\mathcal{N} = 4$ SYM. We find the ratio is crucial

$$\frac{s_{\lambda=\infty}}{s_{\lambda=0}} = \frac{3}{4} \tag{29}$$

since many examples of CFT duals are known in $d = 4$ which have

$$\frac{s_{strong}}{s_{free}} = \frac{3}{4}h \tag{30}$$

where for many theories $\frac{8}{9} \leq h \leq 1.09$.

In the pset you will study the behavior of Wilson loop at a finite T. The physical expectation is: when L is sufficiently large $V(L) \to 0$. This is called "color screening" in QCD. The gravity dual is shown in the following picture, where in small L it is similar as zero temperature but in large L the string connecting two quarks becomes alike two very straight strings hanging from two quarks respectively that looks like two free static quarks.

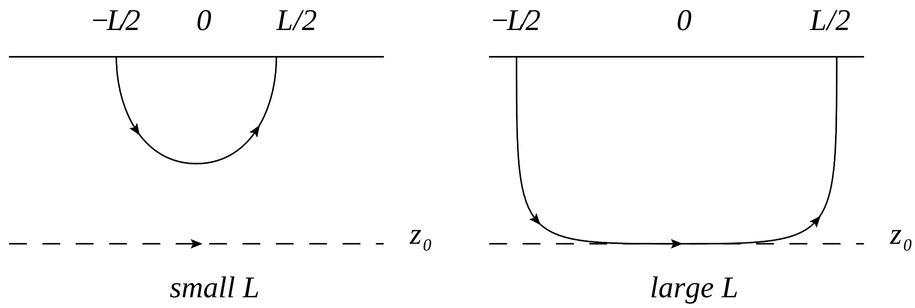

Figure 3: String hanging in AdS black hole