Chapter 2: Deriving AdS/CFT

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Lecture 10

In this chapter, we will focus on:

1. The spectrum of closed and open strings (for gravity and gauge theories).
2. The physics of D-branes – nonperturbative objects in string theory.
3. D-branes as classical gravity (general relativity) solutions.

2.1: PERTURBATIVE (BOSONIC) STRING THEORY

2.1.1: GENERAL SET UP

Consider a string moving in a spacetime $\mathcal{M}$ with the metric ($\mu, \nu = 0, 1, ..., d-1$):

$$ds^2 = G_{\mu\nu}(X) dX^\mu dX^\nu \quad , \tag{1}$$

with the worldsheet Σ parametrizations ($a = 0, 1$):

$$X^\mu(\sigma, \tau) = X^\mu(\sigma^a) \quad . \tag{2}$$

The induced metric on Σ is written as:

$$h_{ab} = G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \quad , \quad ds^2_{ind} = h_{ab}(\sigma, \tau) d\sigma^a d\sigma^b \quad . \tag{3}$$

The string action is defined to be proportional to the area of Σ, written in the following Nambu-Goto form:

$$S_{NG}[X^\mu] = \frac{1}{2\pi\alpha'}\int_\Sigma dA = \frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h} \quad ; \quad [\alpha'] = L^2 \to \alpha' = l_s^2 \quad , \quad T = \frac{1}{2\pi\alpha'} \quad , \tag{4}$$

with l_s is the string length scale (from dimensional analysis) and T is the string tension.

To understand this action, at least at classical level, let's take a look a nearly static string configuration (small fluctuations) extends in X^1-direction in a flat spacetime $\mathbb{R}^{1,d-1}$ and choose the worldsheet parametrizations $\sigma^0 = X^0$ and $\sigma^1 = X^1$ (the remaining coordinates are X^i, with $i = 2, 3, ..., d-1$):

$$\begin{aligned} h_{ab} &= G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu = \begin{pmatrix} -1 + (\partial_0 X^i)^2 & \partial_0 X^i \partial_1 X^i \\ \partial_0 X^i \partial_1 X^i & 1 + (\partial_1 X^i)^2 \end{pmatrix} & (5) \\ \Rightarrow -h &= \Big(1 - (\partial_0 X^i)^2\Big)\Big(1 + (\partial_1 X^i)^2\Big) + (\partial_0 X^i \partial_1 X^i)(\partial_0 X^j \partial_1 X^j) \quad . & (6) \end{aligned}$$

This gives the Nambu-Goto Lagrangian at lowest orders:

$$S_{NG} = \int_\Sigma d^2\sigma \mathcal{L}_{NG} \quad , \quad \mathcal{L}_{NG} = -T + \frac{T}{2}\Big((\partial_0 X^i)^2 - (\partial_1 X^i)^2\Big) + \mathcal{O}\Big((\partial X^i)^4\Big) \quad . \tag{7}$$

From the 0th order, it's straight-forward to see why T can be identified as the string tension (the energy per unit length of the string). The next order shows that the string fluctuations (waves) propagate with the speed of light.

Since the nonpolynomial nature of S_{NG} is inconvenient for calculations, it's much easier to work with the Polyakov's action, which is equivalent to the Nambu-Goto's action at classical level:

$$S_P[\gamma^{ab}, X^\mu] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \quad ; \quad \gamma^{ab} = \gamma^{ab}(\sigma,\tau) \quad . \tag{8}$$

To see this, note that worldsheet stress-energy tensor is defined as:

$$T_{ab} = -4\pi\frac{\delta S_P}{\sqrt{-\gamma}\delta\gamma^{ab}} \quad . \tag{9}$$

Since $\delta S_P = 0$ for variations around the classical solution (on-shell), the equation of motion for γ^{ab} is $T_{ab} = 0$. Using:

$$\delta\sqrt{-\gamma} = -\frac{1}{2}\sqrt{-\gamma}\gamma_{ab}\delta\gamma^{ab} \quad , \tag{10}$$

then the stress-energy tensor can be found:

$$\Big(\int_\Sigma d^2\sigma\Big)^{-1}\delta S_P = -\frac{1}{4\pi\alpha'}\Big(\delta\sqrt{-\gamma}\gamma^{ab}h_{ab} + \sqrt{-\gamma}\delta\gamma^{ab}h_{ab}\Big) = \frac{1}{4\pi\alpha'}\sqrt{-\gamma}\Big(\frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd} - h_{ab}\Big)\delta\gamma^{ab} \tag{11}$$

$$\Rightarrow T_{ab} = \frac{1}{\alpha'}\Big(\frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd} - h_{ab}\Big) = \frac{1}{\alpha'}G_{\mu\nu}\Big(\frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu\partial_d X^\nu - \partial_a X^\mu \partial_b X^\nu\Big) = 0 \quad . \tag{12}$$

This means $\gamma^{ab} = Bh^{ab}$, with $B = B(\sigma,\tau)$ can be arbitrary. Integrate out the worldsheet intrinsic metric field γ^{ab}:

$$S_P[\gamma^{ab} = Bh^{ab}, X^\mu] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma(B^{-1}\sqrt{-h})(Bh^{ab})(h_{ab}) = \frac{1}{2\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h} = S_{NG}[X^\mu] \quad . \tag{13}$$

In quantum field theories and effective field theories, if different theories are related by field redefinitions that preserve symmetries and have the same 1-particle states (Representation Independence Theorem), then the results of observables in spacetime of these theories should agree (although the off-shell process might be different but the on-shell calculations are the same, order by order). Hence it is very tempting to say that S_{NG} and S_P produce the same physics (not only classically but also quantumly), however, the argument is only done on actions of polynomial form. Therefore, one shouldn't think S_{NG} and S_P describe the same physics at quantum level; S_{NG} simply provoke S_P, and the later is much more simple for quantization and computational purposes.

Equation (5) has the form of a 2D field theory of d scalar fields X^μ couples with gravity – the 2D metric γ^{ab} (note that, X^0 field has opposite sign in its kinetic term). Historically, (5) is called a nonlinear σ-model (for S_P, the scalar fields X^μ which takes on values in a nonlinear target manifold Σ). The string path integral quantization will be based on (after taking care of gauge redundancy and adding insertions in ...):

$$\int D\gamma^{ab}DX^\mu e^{iS_P[\gamma^{ab},X^\mu]}\ldots \quad , \tag{14}$$

and this method of quantization has computational advantages as well as conceptual simplicity (especially for a general worldsheet Σ with nontrivial topology), but for the sake of understanding the physical spectrum of strings (the worldsheet describes the noninteracting asymptotic state is topologically trivial), then canonical quantization is more convenient.

The Polyakov Lagrangian:

$$S_P = \int_\Sigma d^2\sigma\mathcal{L}_P \quad . \tag{15}$$

Work with flat spacetime $G_{\mu\nu} = \eta_{\mu\nu}$ ($\mathcal{M} = \mathbb{R}^{1,d-1}$), then the Polyakov action from the 2D worldsheet perspective has 1 set of global symmetry (spacetime Poincare symmetry) and 2 sets of local gauge symmetries (worldsheet diffeomorphism symmetry for reparametrization of the worldsheet and Weyl symmetry for rescaling of the intrinsic metric):

1. Global Poincare transformation (translation and Lorentz rotation):

$$X^\mu(\sigma,\tau) \to X^\mu(\sigma,\tau) + a^\mu \quad ; \quad X^\mu(\sigma,\tau) \to \Lambda^\mu_\nu X^\nu(\sigma,\tau) \quad . \tag{16}$$

2. Local diffeomorphism transformation ($\sigma^a \to \sigma'^a$):

$$X^\mu(\sigma,\tau) \to X'^\mu(\sigma',\tau') = X^\mu(\sigma,\tau) \quad , \quad \gamma^{ab}(\sigma,\tau) \to \gamma'^{ab}(\sigma',\tau') = \frac{\partial\sigma'^a}{\partial\sigma^c}\frac{\partial\sigma'^b}{\partial\sigma^d}\gamma^{cd}(\sigma,\tau) \quad . \tag{17}$$

3. Local Weyl transformation:

$$\gamma^{ab}(\sigma,\tau) \to e^{-2\omega(\sigma,\tau)}\gamma^{ab}(\sigma,\tau) \quad . \tag{18}$$

Note that, conformal symmetry = Diff × Weyl $\Big|_{\gamma_{ab}}$ symmetry.

These symmetries (Poincare and Diff × Weyl) can be used as the guiding principles to (almost) uniquely determined the string action in equation (5). Indeed, for example, in a topological invariant of 2D oriented closed surfaces, the 2D Einstein-Hilbert action also fits the bill:

$$S_\chi[\gamma^{ab}] = \lambda\Big(\frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-\gamma}R\Big) = \lambda\chi(\Sigma) \quad , \quad \chi(\Sigma) = 2-2g \quad . \tag{19}$$

Because of the topology nature, S_χ plays the role similar to the vertex weight in quantum field theory. The closed string coupling is defined to be $g_c = e^\lambda$.

For more generalization, a topological invariant of 2D oriented (both closed and open) surfaces that comes from Einstein-Hilbert action and boundary extrinsic curvature action ($K = \pm t^a n_b \nabla_a t^b$ is the extrinsic curvature of Σ with $t \perp n$ and the sign $\pm$ depends on the type of boundary – timelike: $+$, spacelike: $-$) should be included:

$$S_\chi[\gamma^{ab}] = \lambda\Big(\frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-\gamma}R + \frac{1}{2\pi}\int_{\partial\Sigma} dsK\Big) = \lambda\chi(\Sigma) \quad ; \quad \chi(\Sigma) = 2-2g-b \quad . \tag{20}$$

It's simple to prove this action is diffeomorphism invariant. To see that S_χ is indeed Weyl invariant $\gamma_{ab} \to e^{2\omega}\gamma_{ab}$, note that:

$$\Gamma^a_{bc} \to \Gamma^a_{bc} + \partial_b\omega\delta^a_c + \partial_c\omega\delta^a_b - \partial_d\omega\gamma^{ad}\gamma_{bc} \quad , \quad R \to e^{-2\omega}(R - 2\nabla_a\partial^a\omega) \quad , \quad t \to e^{-\omega}t \quad , \quad n_a \to e^{\omega}n_a \quad , \quad ds \to e^{\omega}ds \quad . \tag{21}$$

Hence:

$$K \to e^{-\omega}(K \mp t^a t_a n^b\partial_b\omega) = e^{-\omega}(K + n^a\partial_a\omega) \quad . \tag{22}$$

Using Stokes theorem:

$$S_\chi \quad \to \quad \lambda\Big(\frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{-\gamma}(R - 2\nabla_a\partial^a\omega) + \frac{1}{2\pi}\int_{\partial\Sigma} ds(K + n^a\partial_a\omega)\Big) \tag{23}$$

$$= \quad S_\chi + \frac{\lambda}{2\pi}\Big(-\int_\Sigma d^2\sigma\sqrt{-\gamma}\nabla_a\partial^a\omega) + \int_{\partial\Sigma} dsn^a\partial_a\omega\Big) = S_\chi \tag{24}$$

Mathematically, the action S_χ only depend on the topological properties of the worldsheet Σ, with $\chi(\Sigma)$ is known as the Euler-characteristic of Σ (g is the number of genus and b is the number of open boundary). 2D gravity is dynamically trivial as S_χ gives no dynamics: indeed, in 2D, from a purely symmetrical fact:

$$R_{abcd} = R_{cdab} = -R_{bacd} = -R_{abdc} \Rightarrow R_{abcd} = \mathcal{R}\varepsilon_{ab}\varepsilon_{cd} \quad . \tag{25}$$

This means the worldsheet Ricci scalar:

$$R = 2\mathcal{R} \quad , \quad R_{ab} = \gamma^{cd}R_{abcd} = \frac{1}{2}R\gamma_{ab} \quad , \tag{26}$$

and the equation of motion for Einstein-Hilbert action (the worldsheet stress-energy tensor $T_{ab} = 0$ on-shell) is satisfied naturally.

Because of the topology nature, S_χ in string theory plays the role similar to the vertex weight in quantum field theory:

1. Add a handle to a closed string worldsheet (creates an extra emission and reabsorption of a closed string) will increase the number of genus by 1 and change the string path weight. In terms of Feynman diagrams this corresponds to the weight of 2 vertices, which can be read-off as the closed string coupling $g_c = e^\lambda$.
2. Add a hole to an open string worldsheet (creates an extra emission and reabsorption of an open string) will increase the number of boundary by 1 and change the string path weight. In terms of Feynman diagrams this corresponds to the weight of 2 vertices, which can be read-off as the open string coupling $g_o = e^{\lambda/2}$.

In oriented string theories, because of a different between Euclidean and Lorentzian signature, the worldsheet Σ can be treated as a Riemann surface (orientable complex manifold) for calculations with imaginary time via analytic continuation (Wick's rotation). For unoriented string theories (eg. type I superstring with orientifold planes), one needs extra works to understand the perturbative regime (eg. worldsheets with cross-caps).

2.1.2: LIGHT-CONE QUANTIZATION

Each physical oscillation mode of a string corresponds to a particle in spacetime. For massless mode, closed string gives a spin 2 particle (graviton) and open string gives a spin 1 particle (gauge particle, like photon or gluon).

The gauge symmetries (Diff $\times$ Weyl) indicate the redundancy in degrees of freedom, and this must be fixed during quantization. For a general topology of the worldsheet Σ, path integral works best for calculation (eg. S-matrix), but for understanding the particle spectrum of the theory by quantizing on a trivial topology of single string propagation (long cylinder for closed string and long sheet for open string), then canonical quantization (old covariant and light-cone quantization) is simpler and faster. It can be shown that these methods of quantization yields the same results for the particle contents [1].

The canonical quantization procedure:

1. Write down the classical equation of motion.
2. Fix the gauge symmetries.
3. Find the complete set of classical solution.
4. Promote classical fields (on worldsheet) to quantum operators, satisfying canonical quantization condition. The classical solutions become solutions to operator equation, and the parameters in classical solutions become creation and annihilation operators.
5. Read-of the spectrum by acting creation operators on the vacuum of the (2D worldsheet) theory.

For single string propagation, closed string (worldsheet topology $S^1 \times \mathbb{R}$) can gets spatial parametrization $\sigma \in [0, 2\pi]$ with:

$$X^\mu(\sigma,\tau) = X^\mu(\sigma,\tau) \quad , \quad \gamma_{ab}(\sigma,\tau) = \gamma_{ab}(\sigma+2\pi,\tau) \quad , \tag{27}$$

while open string (worldsheet topology $L \times \mathbb{R}$) can gets spatial parametrization $\sigma \in [0, \pi]$ with boundary conditions at $\sigma = [0, \pi]$ which can be found by enforcing the nonlocal contribution to δS_P when varying δX^μ goes away:

$$\delta S_P = -\frac{1}{2\pi\alpha'} \int d\tau \sqrt{-\gamma}\, \delta X^\mu \partial_\sigma X_\mu = 0 \tag{28}$$

With this, one can impose at the 2 ends either Dirichlet condition ($\delta X^\mu = 0$) or Neumann condition ($\gamma^{\sigma a}\partial_a X^\mu = 0$). Or in other words, $n^a \partial_a X^\mu = 0$ with $n \perp \partial\Sigma$ (this indeed gives a glimpse of a nonperturbative object in string theory – the D-brane, where strings can end). Let's just look at the Neumann condition (in all directions) for the moment.

Note that, the boundary condition (at the 2 ends of open string) it is not for the sake of simplicity, but for consistency – only for appropriate boundary conditions, can equations of motion be consistently imposed. Indeed, when D-branes are included, the boundary contributions to the stringy dynamics is very important.

The classical equation of motion from (5):

1. For γ^{ab}:

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X^\nu = 0 \tag{29}$$

2. For X^μ:

$$\partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu) = 0 \tag{30}$$

By diffeormophism, the metric can be put in the conformally flat form $\gamma_{ab} = e^{2\omega(\sigma,\tau)}\eta_{ab}$, and Weyl rescaling can be used to get $\gamma_{ab} = \eta_{ab}$ (note that, after this setting one still have a gauge redundancy, called the conformal symmetry).

Indeed, by using the similar trick on a general worldsheet Σ, the whole surface can be cover with a single flat 2D sheet with appropriate holes, parametrizations and identifications (which give rise to the moduli and curvature). This trick is extremely useful for studying string scattering via the path integral quantization, and has a natural generalization to a superworldsheet in superstring.

For example, genus $g = 1$ Riemann surfaces can be built out of a complex plane parametrize by z by cutting-off the regions:

$$|z| \leq (1-\epsilon)\sqrt{|w|} \quad , \quad \frac{1}{1-\epsilon}\frac{1}{\sqrt{|w|}} \leq |z| \quad , \quad \epsilon \ll 1 \quad , \tag{31}$$

and then glueing the annuli:

$$D_0 = \{z_0 \in \mathbb{C} \Big| (1-\epsilon)\sqrt{|w|} < |z_0| < \frac{1}{1-\epsilon}\sqrt{|w|}\} \quad , \quad D_\infty = \{z_0 \in \mathbb{C} \Big| (1-\epsilon)\frac{1}{\sqrt{|w|}} < |z_0| < \frac{1}{1-\epsilon}\frac{1}{\sqrt{|w|}}\} \tag{32}$$

by the transition function:

$$z_0 = f_{,\infty}(z_\infty) = w z_\infty \quad . \tag{33}$$

The complex modulus can be read-off from w.

To get genus $g = 2$ surfaces, firstly cut-off 2 more disks:

$$|z-a| \leq (1-\epsilon)\sqrt{|u|} \quad , \quad |z-b| \leq (1-\epsilon)\sqrt{|u|} \quad , \tag{34}$$

and specify the 2 annular regions D_a and D_b, then glue these together by a transition map $f_{a,b}(z)$. The transition functions $f_{0,\infty}(z)$ and $f_{a,b}(z)$ should be in total depend on 3 complex parameters (up to conformal transformation), which is the 3 complex moduli of this Riemann surfaces.

Higher genus (super)Riemann surfaces, or any other general (super)worldsheet Σ, follow a similar procedure.

With $\gamma_{ab} = \eta_{ab}$, equation (26) can be rewritten:

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = (\partial_\tau^2 - \partial_\sigma^2)X^\mu = (\partial_\tau - \partial_\sigma)(\partial_\tau + \partial_\sigma)X^\mu = 0 \quad . \tag{35}$$

Equation (23) can be further simplified:

$$\begin{aligned} T_{\tau\tau} = T_{\sigma\sigma} &= \frac{1}{2}(\partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu) = 0 \quad . && (36)\\ T_{\tau\sigma} = T_{\sigma\tau} &= \partial_\tau X^\mu \partial_\sigma X_\mu = 0 \quad . && (37) \end{aligned}$$

For open string with Neumann condition, $\partial_\sigma X^\mu(\sigma = 0, \pi; \tau) = 0$.

Note that equation (31) can be derived from the action obtained by directly setting the flat worldsheet metric for the action given in equation (3) (which gives a bunch of free scalar fields). However, quantization of strings is different from merely quantization of $S_P[\gamma^{ab} = \eta^{ab}, X^\mu]$, as one needs to impose the conditions that the stress tensor following from that is 0, which are equation (32) and (33) – the Virasoro constraints (nonlinear constraint equations).

Equation (31) can be immediately solved (x^μ, v^μ can be arbitrary constants):

$$X^\mu(\sigma,\tau) = x^\mu + v^\mu\tau + X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma) \tag{38}$$

For closed strings, X_R^μ and X_L^μ are independent periodic functions of period 2π. For open strings, from the Neumann condition, then at the 2 ends $\partial_\sigma X_R = \partial_\sigma X_L$ as $X_R^\mu = X_L^\mu$ and is periodic in 2π. From here, one could quantize $S_P[\gamma^{ab} = \eta^{ab}, X^\mu]$ first and then impose the Virasoro constraints at quantum level later, or firstly solve these constraints explicitly at classical level and then find unconstrained degrees of freedom hence only quantize these. Light-cone gauge quantization follows the 2nd approach.

After fixing the worldsheet metric, one still have residual gauge freedom (conformal symmetry). Let's introduce:

$$\sigma^{\pm} = \frac{\tau \pm \sigma}{\sqrt{2}} \quad , \quad ds^2 = -2d\sigma^+ d\sigma^- \quad , \tag{39}$$

hence this symmetry can be viewed as the preservation of $\gamma^{ab} = \eta_{ab}$ (up to a Weyl rescaling) as:

$$\sigma^+ \to \tilde{\sigma}^+ = f(\sigma^+) \quad , \quad \sigma^- \to \tilde{\sigma}^- = g(\sigma^-) \quad , \quad ds^2 \to -2\partial_+ f \partial_- g d\sigma^+ d\sigma^- \quad . \tag{40}$$

Since:

$$\tilde{\tau} = \frac{f(\tau+\sigma) + g(\tau-\sigma)}{\sqrt{2}} \Rightarrow \tau \quad , \tag{41}$$

which has the same form as the classical solution of X^μ, then one can fix the gauge completely by choosing appropriate f and g so that:

$$\tau = \frac{X^+}{v^+} \quad , \quad X^{\pm} = \frac{X^0 \pm X^1}{\sqrt{2}} \quad . \tag{42}$$

This is known as the light-cone gauge, as the worldsheet time is fixed by the spacetime light-cone coordinate.

With $X^\mu = (X^+, X^-, X^i)$ (the transverse directions $i = 2, 3, ..., d-1$):

$$dX^\mu dX_\mu = -2dX^+ dX^- + dX^i dX^i \quad . \tag{43}$$

In light-cone gauge, the Virasoro constraints become:

$$\begin{aligned} 2v^+ \partial_\tau X^- &= (\partial_\tau X^i)^2 + (\partial_\sigma X^i)^2 \quad , & (44) \\ v^+ \partial_\sigma X^- &= \partial_\tau X^i \partial_\sigma X^i \quad , & (45) \end{aligned}$$

therefore, X^- can be fully solved in terms of $X^i(\sigma, \tau)$. Therefore, the independence degrees of freedom are X^i. Since X^0 has a "wrong" sign for its kinetic terms, no X^0 in these degrees of freedom actually partly solve a problem of unitarity at quantum level.

Reference

[1] Polchinski, J. (2013). *String Theory, v.1.* Lexington, KY: Cambridge University Press.

8.821 / 8.871 String Theory and Holographic Duality
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