1. Calculate the $\lambda = +\frac{1}{2}$ helicity eigenspinor of an e^- of momentum $\vec{p}\,' = (p\sin\theta, 0, p\cos\theta)$.

This momentum $\vec{p}\,'$ is just a rotation about the y-axis from our standard momentum vector $\vec{p} = (0,0,p)$, for which we found the helicity $\lambda = +\frac{1}{2}$ eigenspinor to be:

$$\frac{1}{2}\vec{\sigma}\cdot\hat{p}\,\chi^{(1)} = \frac{1}{2}\sigma_3\,\chi^{(1)} = +\frac{1}{2}\chi^{(1)}$$

where the spinor is:

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For the e^-, the 4-component Dirac spinor with $\lambda = +\frac{1}{2}$ is:

$$u^{(1)} = N\begin{pmatrix} \chi^{(1)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(1)} \end{pmatrix}$$

this becomes just $\frac{p}{E+m}$ after the dot product.

Now, we want to rotate this about the y-axis to get an eigenspinor corresponding to our mom. vector $\vec{p}\,'$.

To perform this rotation, we can use the relation from eq. (2.26):

$$e^{-i\theta\sigma_2/2} = \cos\frac{\theta}{2} - i\sigma_2\sin\frac{\theta}{2}$$

In (useful) matrix form, this becomes:

$$u(p') = N\begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & & 0 \\ \sin\theta/2 & \cos\theta/2 & & \\ & & c(\frac{\theta}{2}) & -s(\frac{\theta}{2}) \\ & 0 & s(\frac{\theta}{2}) & c(\frac{\theta}{2}) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{bmatrix}$$

$$u(\vec{p}\,') = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \\ \frac{p}{E+m}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \end{bmatrix}$$

5.15 Working in the Dirac-Pauli representation of γ matrices, (5.51), show that at high energies:

$$\gamma^5 u^{(s)} \simeq \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} u^{(s)} \qquad (*)$$

where $u^{(s)}$ is the e^- spinor of (5.27). That is, show that in the extreme relativistic limit, the chirality operator (γ^5) is equal to the helicity operator; and so, for example, $\frac{1}{2}(1-\gamma^5)u = u_L$ corresponds to an e^- of negative helicity

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The Dirac-Pauli matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad (5.51)$$

And the e^- spinor:

$$u^{(s)} = N\begin{bmatrix} \chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{(s)} \end{bmatrix} \qquad \text{where: } \chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (5.27)$$

In the relativistic case, $p \gg m$ thus $E^2 = p^2 + m^2$ becomes $E^2 \doteq p^2$

And our spinor becomes,

$$u^{(s)} \cong N\begin{bmatrix} \chi^{(s)} \\ (\vec{\sigma}\cdot\hat{p})\chi^{(s)} \end{bmatrix}$$

The action of γ^5 on this is:

$$\gamma^5 u^{(s)} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} \chi^{(s)} \\ (\vec{\sigma}\cdot\hat{p})\chi^{(s)} \end{bmatrix} = \begin{bmatrix} (\vec{\sigma}\cdot\hat{p})\chi^{(s)} \\ \chi^{(s)} \end{bmatrix} \qquad (**)$$

Comparing this to the RHS of eq'n ($*$):

$$\begin{bmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{bmatrix}\begin{bmatrix} \chi^{(s)} \\ (\vec{\sigma}\cdot\hat{p})\chi^{(s)} \end{bmatrix} = \begin{bmatrix} (\vec{\sigma}\cdot\hat{p})\chi^{(s)} \\ (\vec{\sigma}\cdot\hat{p})^2\chi^{(s)} \end{bmatrix} = \begin{bmatrix} (\vec{\sigma}\cdot\hat{p})\chi^{(s)} \\ \chi^{(s)} \end{bmatrix}$$

which is the same as eq'n ($**$)!

I show this on p. (2.2)

5.15, cont'd

What is $(\vec{\sigma}\cdot\hat{p})^2$?

Generally, $\vec{\sigma}\cdot\hat{p}$ is:

$$\vec{\sigma}\cdot\hat{p} = \frac{p_1}{p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{p_2}{p}\begin{bmatrix} 0 & -i \\ +i & 0 \end{bmatrix} + \frac{p_3}{p}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{1}{p}\begin{bmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{bmatrix}$$

Square this:

$$(\vec{\sigma}\cdot\hat{p})^2 = \frac{1}{p^2}\begin{bmatrix} p_3^2 + p_1^2 + p_2^2 & 0 \\ 0 & p_1^2 + p_2^2 + p_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. In γ^5 diagonal representation $\quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\gamma^5 \begin{pmatrix} \xi_+ \\ 0 \end{pmatrix} = + \begin{pmatrix} \xi_+ \\ 0 \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix}$$

$$\gamma^5 \begin{pmatrix} 0 \\ \chi_- \end{pmatrix} = - \begin{pmatrix} 0 \\ \chi_- \end{pmatrix}$$

ξ_+ & χ_- are $\pm$ eigenstates of γ^5, chirality.

The Dirac Eqs in terms of ξ_+ & χ_- are:

$$\vec{\sigma}\cdot\hat{p}\,\xi_+ = \xi_+ + \frac{m}{E}\chi_- \quad \text{—} ①$$

$$\vec{\sigma}\cdot\hat{p}\,\chi_- = -\chi_- - \frac{m}{E}\xi_+ \quad \text{—} ②$$

① $+ \frac{m}{2E}$ ② yields

$$\vec{\sigma}\cdot\hat{p}\left(\xi_+ + \frac{m}{2E}\chi_-\right) = \left(\xi_+ + \frac{m}{2E}\chi_-\right) + \mathcal{O}\left(\frac{m^2}{E^2}\right)$$

② $+ \frac{m}{2E}$ ① yields

$$\vec{\sigma}\cdot\hat{p}\left(\chi_- + \frac{m}{2E}\xi_+\right) = -\left(\chi_- + \frac{m}{2E}\xi_+\right) + \mathcal{O}\left(\frac{m^2}{E^2}\right)$$

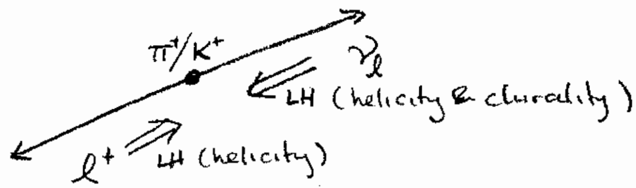
So helicity eigenstates are

$$\xi_+ + \frac{m}{2E}\chi_- \quad \text{for } h = +$$

$$\chi_- + \frac{m}{2E}\xi_+ \quad \text{for } h = -1$$

4. Use the results from #3 to obtain the ratio of the branching ratio of $\pi^+ \to e^+ \nu_e$ to that of $\pi^+ \to \mu^+ \nu_\mu$. And for the ratio of BF's for $K^+ \to e^+ \nu_e$ to $K^+ \to \mu^+ \nu_\mu$.

Because the π & K are both pseudoscalar particles, they have spin-0. Hence, (assuming $m_\nu = 0 \Rightarrow$ helicity & chirality states are the same), we have a decay like:



Because it's a weak decay, the ν must have LH chirality (& the $\nu_{L(\text{chiral})}$ has LH helicity).

To get a spin-0 wavefcn (in accordance w/ our initial state) we need the charged lepton, ℓ^+, to also have LH helicity. However, b/c the weak interaction couples to the LH chirality state, if chirality & helicity eigenstates were equal (ie. if $m=0$), this would give us only RH (helicity) ℓ^+'s (b/c LH chirality = RH helicity for ($m=0$) anti-fermions).

Fortunately, we found (in #3) that a LH chirality spinor, ξ_+, has both h_+ & h_- states. Here, we want eq'n (4) from p. 3.2:

$$h_+ = (\vec{\sigma}\cdot\hat{p})\xi_+ + \frac{1}{2}\frac{m}{E}\chi_- \qquad (4)$$

[This was for a fermion, so for our ℓ^+, this actually describes the h_- state (LH helicity anti-fermion].

The first term is zero b/c the weak interaction does not couple to RH chirality (ξ_+). Thus, for our ℓ^+ we have:

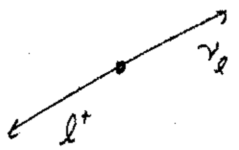
$$\underset{\uparrow \text{anti-fermion!}}{h_-} = \frac{1}{2}\frac{m}{E}\chi_-$$

4, cont

Now, we see quite simply that the ratio of BF's is given by:

$$R_\pi \equiv \frac{BF(\pi^+ \rightarrow e^+ \nu_e)}{BF(\pi^+ \rightarrow \mu^+ \nu_\mu)} = \frac{\left(\frac{1}{2}\frac{m_e}{E_e}\right)^2}{\left(\frac{1}{2}\frac{m_\mu}{E_\mu}\right)^2} \qquad (\mathrm{I})$$

The energies, E_e & E_μ are given by (generally, E_ℓ):



Energy cons. $\quad m_\pi = E_\ell + E_\nu \qquad (*)$

Mom. cons. $\quad p_\ell = p_\nu \qquad (**)$

for the massless ν, $E_\nu = p_\nu$. In $(**) \Rightarrow p_\ell = E_\nu$
using this in $(*)$:

$$m_\pi = E_\ell + p_\ell$$
$$m_\pi - p_\ell = \sqrt{m_\ell^2 + p_\ell^2}$$
$$(m_\pi - p_\ell)^2 = m_\ell^2 + p_\ell^2$$
$$m_\pi^2 + \cancel{p_\ell^2} - 2m_\pi p_\ell = m_\ell^2 + \cancel{p_\ell^2}$$
$$p_\ell = \frac{m_\pi^2 - m_\ell^2}{2m_\pi} = \frac{1}{2}m_\pi - \frac{m_\ell^2}{2m_\pi}$$

plugging this into $(*)$:

$$E_\ell = m_\pi - \overset{= E_\nu}{p_\ell}$$
$$\boxed{E_\ell = \frac{1}{2}m_\pi + \frac{m_\ell^2}{2m_\pi}} \qquad = \frac{m_\pi^2 + m_\ell^2}{2m_\pi}$$

(this is true for Kaons, as well
(substituting m_K for m_π)

This makes eq'n (I) become:

$$R_\pi = \left(\frac{m_e^2}{m_\mu^2}\right)\left(\frac{E_\mu}{E_e}\right)^2$$
$$= \left(\frac{m_e^2}{m_\mu^2}\right)\left(\frac{m_\pi^2 + m_\mu^2}{m_\pi^2 + m_e^2}\right)^2$$

4, cont

The ratio of energies:

$$R_{E\pi} = \left(\frac{E_\mu}{E_e}\right)^2 = \left(\frac{m_\pi^2 + m_\mu^2}{m_\pi^2 + m_e^2}\right)^2 = \frac{m_\pi^4 + m_\mu^4 + 2m_\pi^2 m_\mu^2}{m_\pi^4 + m_e^4 + 2m_\pi^2 m_e^2}$$

$$= \frac{1 + 2\frac{m_\mu^2}{m_\pi^2} + \frac{m_\mu^4}{m_\pi^4}}{1 + 2\frac{m_e^2}{m_\pi^2} + \frac{m_e^4}{m_\pi^4}} \sim 3 \qquad (\mathrm{II})$$

So we expect

$$\mathcal{R}_\pi \doteq 3\frac{m_e^2}{m_\mu^2} \simeq 3\left(2.3\times10^{-5}\right)$$

Calculating exactly, (II) gives:

$m_\mu = 105.66$ MeV

$m_\pi = 139.57$ MeV

$m_e = 0.511$ MeV

$m_K = 493.68$ MeV

$R_{E\pi} = 2.475$

$R_{EK} = 1.094$

Thus, the results for π^+ & K^+ decays:

$$\frac{BF(\pi^+ \to e^+\nu_e)}{BF(\pi^+ \to \mu^+\nu_\mu)} = (2.475)\left(\frac{m_e}{m_\mu}\right)^2 \simeq 5.8\times10^{-5}$$

$$\frac{BF(K^+ \to e^+\nu_e)}{BF(K^+ \to \mu^+\nu_\mu)} = (1.094)\left(\frac{m_e}{m_\mu}\right)^2 = 2.6\times10^{-5}$$

From the PDG, I find:

$\mathcal{R}_\pi = 1.23\times10^{-4}$ ~ a factor of 2 different from my calculation. (*)

$\mathcal{R}_K = 2.4\times10^{-5}$

(*) I believe that this error comes from letting the ratio be simply $\left(\frac{M}{E}\right)^2$; in this assumption, I have stated that this is all that differs b/w the 2 decays – but in truth, the ν energy spectrum is also different, so this assumption is not quite accurate.

Using the amplitude (12.47) and integrate over the phase space (12.52), one get the width

$$\Gamma(\pi \to l\bar{\nu}) = \frac{G^2}{8\pi} f_\pi^2 m_\pi m_l^2 \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2 \qquad \begin{array}{c} l = \mu \text{ or } e \\ (12.54) \end{array}$$

Thus

$$\frac{\Gamma(\pi \to \mu\bar{\nu})}{\Gamma(\pi \to e\bar{\nu})} = \left(\frac{m_\mu}{m_e}\right)^2 \left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2 - m_e^2}\right)^2 = \frac{1}{1.2 \times 10^{-4}}$$