PROFESSOR: OK, let's start. So, we've moved onto the two-dimensional xy model. This is a system where, let's say on each side of a square lattice you put a unit vector that has two components and hence, can be described by an angle theta. So basically, at each side, you have an angle theta for that side.

And there is a tendency for a neighboring spins to be aligned and the partition function can be written as a sum over all configurations, which is equivalent to integrating over all of these angles. And the weight that wants to make the two neighboring spins to be parallel to each other. So we have a sum over nearest neighbors. And the dot product of the two spins amounts to looking at the cosine of theta i minus theta j. And so, we have this factor over here.

Now, if we go to the limit where k is large, then the cosine will tend to keep the angles close to each other. And we are tempted to expand this around the configurations where everybody's parallel. Let's call that nk by the factor of 2. And then, expanding with the cosine to the next order, you may want to replace this-- let's call this k0-- and have a factor of k, which is proportional to k0 after some lattice spacing. And integral of gradient of theta squared.

So basically, the difference between the angles in the continuum version, I want to replace with the term that tries to make the gradient to be fixed. OK. Now the reason I put these quotes around the gradient is something that we noticed last time, which is that in principal, theta is defined up to a multiple of 2 pi. So that if I were to take a circuit along the lattice that comes back to itself. And all along, this circuit integrate this gradient of theta, so basically, gradient of theta would be a vector. I integrated along a circuit. And by the time I have come back and close the circuit to where I started, the answer may not come back to 0. It may be any integer

multiple of 2 pi. All right.

So how do we account for this? The way we account for this is that we note that this gradient of data I can decompose into two parts. One, where I just write it as a gradient of some regular function. And the characteristic of gradient is that once you go over a closed loop and you integrate, you essentially are evaluating this field phi at the beginning and the end. And for any regular single valued phi, this would come back to zero. And to take care of this fact the result does not have to come to zero if I integrate this gradient of theta, I introduce another field, u, that takes care of these topological defects. OK?

So that, really, I have to include both configurations in order to correctly capture the original model that had these angles. OK, so what can this u be? We already looked at what u is for the case of one topological defect. And the idea here was that maybe I had a configuration where around a particular center, let's say all of the spins were flowing out, or some other such configuration such that when I go over a large distance r from this center, and integrate this field u, just like I did over there, the answer is going to be, let's say, 2 pi n. So there's this u. And I integrated along this circle. And the answer is going to be 2 pi n. Well, clearly, the magnitude of u times 2 pi r, which is the radius of the circle is going to be 2 pi times some integer-- could be plus, minus 1, plus minus 2. And so, the magnitude of u is n over r.

The direction of u is orthogonal to the direction of r. And how can I show that? Well, one way I can show that is I can say that it is z hat crossed with r hat, there z hat is the vector that comes out of the plane. And r hat is the unit vector in this direction. u is clearly orthogonal to r. The direction of the gradient of this angle is orthogonal to r. It is in the plane so it's orthogonal to this. And this I can also write as z hat 3 crossed with the gradient of log of r, with some cut-off. Because the gradient of log of r will give me, essentially, 1 over R in the direction of r hat. And this is like the potential that I would have for a charge in two dimensions, except that I have rotated it by somewhat. And this I can also write as minus the curl of z hat log r over a with a factor of n.

And essentially, what you can see is that the gradient of data for a field that has this topological defect has a part can be written as a potential gradient of some y, and a part that can be written as curl to keep track of these vortices, if you like. If you were to think of this gradient of data like the flow field that you would have in two dimensions. It has a potential part. And it has a part that is due to curvatures and vortices, which is what we have over here. OK.

So this is, however, only for one topological defect. What happens if I have many such defects? What I can do is, rather than having just one of them, I could have another topological defect here, another one here, another one there. There should be a combination of these things. And what I can do in order to get the corresponding u is to superimpose solutions that correspond to single ones. As you can see that this is very much like the potential that I would have for a charge at the origin, and then taking the derivative to create the field. And you know that as long as things are linear, and there aren't too many of them, you can superimpose solutions for different charges. You could just add up the electric fields.

So what I'm claiming is that I can write u as minus n curl of z hat, times some potential u of r, where psi of r is essentially the generalization of this log. I can write it as a sum over all topological defects. And I will have the n of that topological defect times log of r minus ri divided by a, where ri are the locations of these. So there could be a vortex here at r1 with charge n1, and other topological defect here at r2 with charge n2, and so forth. And I can construct a potential that basically looks at the log of r minus ri for each individual one, and then do this. OK?

I will sometimes write this in a slightly different fashion. Recall that we had the Coulomb potential, which was related to log by just a factor of 1 over 2 pi. So the correct version of defining the Coulomb potential is this. So this I can write as the Coulomb potential, provided that I multiply the 2 pi ni. And I sometimes will call that qi. So essentially, qi is 2 pi and i is the charge of the topological defect. It can be plus or minus 2 pi. And then, the potential is constructed by constructing superposition of those charges divided or multiplied with appropriate Coulomb potential. OK?

All right. So I can construct a cost for creating a configuration now. Previously, I had this integral gradient of theta squared in the continuum. And my gradient of theta squared has now a part that is the gradient of a regular, well-behaved potential, and a part that is this field u, which is minus-- oops. Then I don't need the ni's because I put the ni as part of the psi. Curl of z hat psi of r. So phi is a regular function. Psi with curl will give me the contribution of the topological defect involves both the charges and the positions of these topological defects. OK?

And this whole thing has to be squared, of course. This is my gradient squared. And if I expand this, I will have three terms. I have a gradient of this 5 squared. I have a term, which is minus 2 gradient of phi dot producted with curl of z hat psi. And I have a term that is curl of z hat psi squared. OK? Again, if you think of this as vector, this is a vector whose components are the xy and the y phi, whereas this is a vector whose components are, let's say, dy psi minus dx psi. Because of the curl operation-- the x and y components-- one of them gets a minus sign. Maybe I got the minus wrong, but it's essentially that structure.

Now you can see that if I were to do the integration here, there is a dx psi, dy psi. I can do that integration by parts and have, let's say phi dx dy psi. And then, I can do the same integration by parts here. And I will have phi minus dx dy psi. So if I do integration by part, this will disappear. Another way of seeing that is that the gradient will act on the curl. And the gradient of the curl of a vector is 0, or otherwise, the curl will act on the gradient with [INAUDIBLE]. So basically, this term does not contribute. And the contribution of this part and the part from topological defects are decoupled from each other. So there's essentially the Gaussian type of stuff that we calculated before is here. On top of that, there is this part that is due to these topological defects.

Again, this vector is this squared. You can see that if I square it, I will get dy psi squared plus dx psi squared. So to all intents and purposes, this thing is the same thing as a gradient of psi squared. Essentially, gradient of psi and curl of psi are the same vector, just rotated by 90 degrees. Integrating the square of one over the whole space is the same as integrating the square of the other. OK?

So now, let's calculate this contribution and what it is. Integral d2 x gradient of psi squared with K over 2 out front-- actually you already know that. Because psi we see is the potential due to a bunch of charges. So this is essentially the electric field due to this combination of charges integrated over the entire space. It's the electrostatic image. But let's go through that step by step.

Let's do the integration by parts. So this becomes minus k over 2. Integral d2 x psi the gradient acting on this will give me Laplacian of psi. Of course, whenever I do integration by parts, I have to worry about boundary terms. And essentially, if you think of what you will be seeing at the boundary, far, far away from there all of these charges are, let's say. Essentially, you will see the electric field due to the combination of all of those charges.

So for a single one, I will have a large electric field that will go as 1 over r. And we saw that integrating that will give me the log. So that was not particularly nice. So similarly, what these boundary terms would amount to would give you some kind of a logarithmic energy that depends on the next charge that you have enclosed. And you can get rid of it by setting the next charge to be zero. So essentially, any configuration in which the sum total of our topological charges is non-zero will get a huge energy cost as we go to large distances from the self-energy, if you like, of creating this huge monopole. So we are going to use this condition and focus only on configurations that are charged topological charge neutral.

OK. Now our psi is what I have over here. It is sum over i qi. This Coulomb interaction-- r minus ri. And therefore, Laplacian of psi is essentially taking the Laplacian of this expression is the sum over j qu. Laplacian of this is the delta function. So basically, that was the condition for the Coulomb potential. Or alternatively, you take 2 derivative of the log, and you will generate the delta function. OK?

So, what you see is that you generate the following. You will get a minus k over 2 sum over pairs i and j, qi, qj. And then I have the integral over x or r-- they're basically the same thing. Maybe I should have written this as x. And the delta

function insures that x is set to be the other i. So I will get the Coulomb interaction between ri minus rj. So basically, what you have is that these topological defects that are characterized by these integers n, or by the charges 2 pi n, have exactly this logarithmic Coulomb interaction. in two dimensions. And as I said, this thing is none other than the electrostatic energy. The electrostatic energy you can write either as an integral of the electric field squared. Or you can write as the interaction among the charges that give rise to that electric field. OK?

So, what I can do is I can write this as follows. First of all, I can maybe re-cast it in terms of the n's. So I will have 2 pi ni, 2 pi nj. So I will get minus 4pi squared k. There's a factor of one-half. But this is a sum over i and j-- so every pair is now counted twice. So I get rid of that factor of one-half by essentially counting each pair only once. So I have the Coulomb interaction between ri and rj, which is this 1 over 2 pi log of ri minus rj with some cut-off.

And then, there's the term that corresponds to i equals 2j. So I will have a minus, let's say, 4pi squared k sum over i. And I forgot here to put n, i, and j. I will have ni squared. The Coulomb interaction at zero. ri equals [INAUDIBLE].

Now clearly, this expression does not make sense. What it is trying to tell me is that there is a cost to creating one of these topological charges. And all of this theory-- again, in order to make sense, we should remember to put some kind of a short distance cut-off a. All right? And basically, replacing this original discrete lattice with a continuum will only work as long as

I keep in mind that I cannot regard things at the level of lattice spacing, and replace it by that formula, as we saw, for example, here. If I want to draw a topological defect, I would need right at the center to do something like this-- where replacing the cosines with the gradient squared kind of doesn't make sense. So basically, what this theory is telling me is that once you get to a very small distance, you have to keep track of the existence of some underlying lattice and the corresponding things. And what's really this is describing for you is the core energy of creating a defect that has object ni.

OK. What do I mean by that is that over here, I can calculate what the partition function is for one defect. This we already did last time around. And for that, I can integrate out this energy that I have for the distortions. It's an integral of n over r squared. And this integration gave me this factor of e to the minus pi k log of r over a-- actually, that was 2 pi k.

Actually, let's do this correctly once. I should have done it earlier, and I forgot. So you have one defect. And we saw that for one defect, the field at the distance r has n over r in magnitude. And then, the net energy cost for one of these defects-- if I say that I believe this formula starting from a distance a is the k over 2 integral from a. Let's say all the way up to the size of my system. I have 2 pi r dr from a shell at radius r magnitude of this u squared. So I have n squared over r squared.

But then, I have to worry about all of the actual things that I have off the distance a. So on top of this, there is a core energy for creating this object that certainly explicitly depends on where I sit this parameter a. OK? This part is easy. It simply gives me pi km squared. And then I have the integral of 1 over r, which gives me log of L over a.

So if I want to imagine what the partition function of this is-- one defect in a system of size L-- I would say that z of one defect is Boltzmann weight responding to creating this entity. So I have e to the minus pi k m squared log of L over a. And then I have the core energy that corresponds to this. And then, as we discussed, I can place this anywhere in the system if I'm calculating the partition function. So there's an integration over the position of this that is implicit. And so, that's going to give me the square of the size of my system, except that I am unsure as to where I have placed things up to this cut-off a.

So really, the number of distinct positions that I have scales like L over a squared. So the whole thing we can see scales like L over a 2 minus pi km squared. And then, there is this factor of e to the minus this core energy evaluated at the distance a that I will call y. Because again, in some sense, there's some arbitrariness in where I choose a. So this y would be a function of a, if we depend on that choice.

But the most important thing is that if I have a huge system, whether or not this partition function, as a function of the size of the system, goes to infinity or goes to 0 is controlled by this exponent 2 minus pi k. Let's say we focus on the simplest of topological defects corresponding to n equal 2 minus plus 1. You expect that there is some potentially critical value of k, which is 2 over pi, that distinguishes the two types of behavior. OK?

But this picture is nice, but certainly incomplete. Because who said that there's any legitimacy in calculating the partition function that corresponds to just a single topological defect. If I integrate over all the configurations of my angle field, I should really be doing something that is analogous to this and calculating a partition function that corresponds to many defects. And actually what I calculated over here was in some sense the configuration of spins given that there is a topological defects that has the lowest energy. Once I start with this configuration-- let's say, with everybody radiating out-- I can start to distort them a little bit which amounts to adding this gradient of phi to that.

So really, the partition function that I want to calculate and wrote down at the beginning-- if I want to calculate correctly, I have to include both these fluctuations and these fluctuations corresponding to an arbitrary set of these topological defects. And what we see is that actually, the partition functions and the energy costs of the two components really separate out. And what we are trying to calculate is the contribution that is due to the topological defects. And what we see is that once I tell you where the topological defects are located, the partition function for them has an energy component that is this Coulomb interaction among the defect.

But there is a part that really is a remnant of this core energy that we were calculating before. So when I was sort of following my nose here, I had forgotten a little bit about the short distance cut-off. And then, when I encountered this C of zero, it told me that I have to think about the limit when two things come close to each other. And I know that that limit is constrained by my original lattice, and more importantly, by the place where I am willing to do self-averaging and replace this sum with a gradient.

OK, so basically, this is the explanation of this term. So the only thing that we have established so far is that this partition function that I wrote down at the beginning gets decomposed into a part that we have calculated before, which was the Gaussian term, and is caused, really, the contribution due to spin waves. So this is when we just consider these [INAUDIBLE] modes, we said that essentially you can have an energy cost that is the gradient squared. So this is the part that corresponds to integral d phi into the minus k over 2 integral d2 x gradient of phi squared, where phi is a well-behaved, ordinary function.

And what we find is that the actual partition function also has the contribution from the topological defects. And that I will indicate by ZQ. And Q stands for Coulomb gas. Because this partition function Z sub Q is like I'm trying to calculate this system of degrees of freedom that are characterized by charges n that can be anywhere in this two-dimensional space. And the interaction between them is governed by the Coulomb interaction in two dimensions. So to calculate this, I have to sum over all configuration of charges. The number of these charges could be zero, could be two, could be four, could be six, could be any number. But I say even numbers because I want to maintain the constraint of neutrality. Sum over ni should be zero.

So I want to do that constrained sum. So I only want to look at neutral configurations. Once I have specified-- let's say that I have eight charges-- four plus and four minus-- well, there is a term that is going to come from here and I kind of said that the exponential of this term I'm going to call y. So I have essentially y raised to the power of the number of charges. Let's call this sum over i ni squared. And I'm actually just going to constrain ni to be minus plus 1. I'm going to only look at these primary charges. So the sum over i and i squared is just the total number of charges irrespective of whether they are plus or minus. It basically is replacing this.

And then, I have to integrate over the positions of these charges. Let's call this total number n. So I have to integrate i1 2n d2 xi the position of where this charge is and then interaction which is exponential of minus 4phi squared k sum over i less than j. And i and j the Coulomb interaction between location let's say xi and xj.

Actually, I want to also emphasize that throughout, I have this cut-off. So when I was integrating over one, I said that the number of positions that I had was not L squared, but L over a squared to make it dimensionless. I will similarly make these interactions dimensioned as I divide by a squared. And so basically, this is the more interesting thing that we want to calculate. Also, again, remember I wrote this a squared down here, also to emphasize that within this expression, the minimal separation that I'm going to allow between any pair of charges is off the order of a. I have integrated out or moved into some continuum description any configuration in which the topological charges are less than distance a. OK? Yes?

AUDIENCE: Essentially, when we were during [INAUDIBLE] it was canonical potential, [INAUDIBLE], canonical ensemble. And this is more like grand canonical ensemble?

PROFESSOR: Yes. So, as far as the original two-dimensional xy model is concerned, I'm calculating a canonical partition function for this spin or angle degrees of freedom. And I find that that integration over spin angle degrees of freedom can be decomposed into a Gaussian part and a part that as you correctly point out corresponds to a grand canonical system of charges. So the number of charges that are going to appear in the system I have not specified whether it is determined implicitly by how strong these parameter was.

AUDIENCE: [INAUDIBLE] of canonical potential?

PROFESSOR: y plays the role of E to the beta mu. The quantity that in 8333 we were writing as z-- E to the beta mu small z. OK?

So, we thought we were solving the xy model. We ended up, indeed, with this grand canonical system, which is currently parametrized by two things. One is this k, which is this strength of the potential. The other is this y. Of course, since this system originally came from an xy model that went only one parameter, I expect this y to also be related to k. But just as an expression, we can certainly regard it as a system that is parametrized by two things-- the k and the y. For the case of the xy model, there will be some additional constraint between the two.

But more generally, we can look at this system with its two parameters. And essentially, we will try to make an expansion in y. You'll say that, OK, presumable, I know what is going to happen when y is very, very small. Because then, in the system I will create only a few charges. If I create many charges, I'm going to penalize by more and more factors of y. So maybe through leading order, the system would be free of charge. And then, there would be a few pairs that would appear here and there. In fact, there should be a small density of them, even no matter how small I make y. There will be a very small density of these things that will appear.

And presumably, these things will always appear close to each other. So I will have lots and lots of these pairs-- well, not lots and lots of these pairs-- a density of them that is controlled by how big y is. And as I make y larger-- so this is y becoming larger-- then presumably, I will generate more and more of these pairs. And once I have more and more of these pairs, they could, in principle, get into each other's way. And when they get into each other's way, then it's not clear who is paired with whom. And at some point, I should trade my picture of having a gas of pairs of these objects to a plasma of charges, plus and minus, that are moving all over the place. So as I tune this parameter y, I expect my system to go from a low density phase of atoms of plus-minus bound to each other to a high density phase where I have a plasma of plus and minuses moving all over the place. Yes?

AUDIENCE: So, y is related to the core energy.

PROFESSOR: Yes.

AUDIENCE: And core energy is defined through [INAUDIBLE] direction at zero separation--

PROFESSOR: Well, no. Because the Coulomb description is only valid large separations. When I get to short distances, who knows what's going on? So there is some underlying microscopic picture that determines what the core energy is. Very roughly, yes, you would expect it to have a form that is of e to the minus k with some coefficient that comes from adding all of those interactions here. Yes?

AUDIENCE: Just based on the sign convention, you're saying if you increase or decrease y, that it will go from a low density--

PROFESSOR: OK, so y is the exponential of something. y equals to zero means I will not create any of these things. y approaching 1-- I will create a lot of them. There's no cost at y equals to one. There's no core energy. I can create them as I want.

AUDIENCE: So this would be like y equals minus epsilon c. Is that right?

PROFESSOR: Yeah. Didn't I have that? You see in the exponential it is with the minus. OK. But in any case, that is the expectation. Right? So I expect that when I calculate, I create one of these defects. There is an energy cost which is mostly from outside. And then, there's an additional piece on the inside. So the exponential of that additional piece would be a number that is less than 1.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah. I mean, the original model has some particular form. And actually, the interactions of the original model, I can make more complicated. I can add the full spin interaction, for example. It doesn't affect the overall form much, just modifies what an effective k is, and what the core energy is independent. OK? All right.

But the key point is that this system potentially has a phase transition as you change the parameter of y. And another way of looking at this transition is that what is happening here in different languages, you can either call it insulator or a dielectric. But what is happening here in different languages, you can either call, say, a metal or, as I said, maybe a plasma.

The point is that here you have free charges. Here you have bound pairs of charges. And they respond differently to, let's say, an external electromagnetic field. So once we have this picture, let's kind of expand our view. Forget about the xy model. Think of a system of charges. And notice that in this low-density phase, it behaves like a dielectric in the sense that there are no free charges. And here, there will be lots of mobile charges. And it behaves like a metal.

What do I mean by that? Well, here, if I, let's say, bring in an external electric field. Or maybe if I put a huge charge, what is going to happen is that opposite charges will accumulate. Or there will be, essentially, opposite charges for the field. So that once you go inside, the fact that you have an external electric field or a charge is completely screen. You won't see it. Whereas here, what is going to happen is that if you put in an electric field, it will penetrate into the system although it will be weakened a little bit by the re-orientation of these charges. Now, if you put a plus charge, the effect of that plus charge would be felt throughout, although weakened a little bit. Because again, some of these dipoles will re-orient in that. OK?

So, this low-density phase we can actually try to parametrize in terms of a weakening of the interactions through a dielectric constant epsilon. And so, what I'm going to try to calculate for you is to imagine that I'm in the limit of low density or small y and calculate what the weakening is, what the dielectric function is, perturbatively in y. Yes?

AUDIENCE: If you were talking about the real electric charges and the way to act on that [INAUDIBLE] real electric field or charge. But if we are talking about topological charges, what would be kind of conjugate force to that?

PROFESSOR: OK. It's not going to be easy. I have to do something about say, re-orienting all of the spins on the boundaries, et cetera. So let's forget about that. The point is that mathematically, the problem is reduced to this system. And I can much more easily do the mathematics if I change my perspective and think about this picture. OK? And that's the thing you have to do in theoretical physics. You basically take advantage of mappings of one model to another model in order to refine your intuition using some other picture. So that's what we are going to do.

So completely different picture from the original spin models-- imagine that you have indeed a box of this material. And this box of material has, because you're wise, more some combination of these plus and minus charges in it. And then, what I do is that I bring externally a uniform electric field in this direction. And I expect that once inside the material, the electric field will be reduced to a smaller value that I will

call E prime because of the dielectric function.

Now, if you ever calculated dielectric functions, that's exactly what I'm going to do now. It's a simple process. What you do, for example, is you do the analog of Gauss' theorem. Let's imagine that we draw a circuit such as this that is partly on the inside, and partly on the outside. So I can calculate what the flux of the electric field is through this circuit, the analog of the Gaussian pillbox. And so, what I have is that what is going on is E. If I call this distance to be L, the flux integrated through the entire thing is E minus E prime times f. So this is the integral of the divergence of the electric field . And by Gauss' theorem, this has to be charge enclosed inside.

OK. Now, why should there be any charge enclosed inside when you have a bunch of plus and minuses. I mean, there will be some pluses and minuses out here as I have indicated. There will be some pluses and minuses that are inside. But the net of these would be zero. So the only place that you get a net charge is those dipoles that happen to be sitting right at the boundary. And then, I have to count how many of them are inside. And some of them will have the plus inside. And some of them will have the minus inside.

And then, I have to calculate the net. The thing is that my dipoles do not have a fixed size. The size of these plus/minus molecules r can be variable itself. OK? So there will be some that are tightly bound to each other. There may be some that are further apart, et cetera. So let's look at pairs that are at the distance r and ask how many of them hit this boundary so that one of them would be inside, one of them will be outside. OK? So, that number has to be proportional to essentially this area. What is that area? On one side, it is L. On the other side, it is R. But if the dipole is oriented at an angle theta, it is, in fact r cosine theta. OK?

So that's the number. Now, what I will have here would be the charge 2 pi. So this is qi. Actually, it could be plus or minus. The reason that there's going to be more plus as opposed to minus is because the dipole gets oriented by the electric field. So I will have a term here that is E to the E prime times q ir-- so that's 2 pi r. So this is qr again, times cosine of theta. So we can see that, depending on cosine of theta

being larger than pi or less than pi, this number will be positive or negative. And that's going to be modified by this number also. And of course, the strength of this whole thing is set by this parameter k. And also how likely it is for me to have created a dipole of size r is controlled by precisely this factor. A dipole is something that has two cores. So it is something that will appear at order of y squared. And there is the energy, according to this formula, of separating two things. And so you can see that essentially, n of r-- maybe I will write it separately over here-- is y squared times E to the minus 4pi squared k. And from here, I have log of r divided by a. And then there's a factor of 2 pi because the Coulomb potential is this.

So, this is going to be y squared a over r to the power of 2 pi k. The further you try to separate these things, the more cost you have to pay. OK. So if you were trying to calculate the contribution of, say, polarizable atoms or dipoles to the dielectric function of a solid, you would be doing exactly this same calculation. The only difference is that the size of your dipole would be set by the size of your molecule and ultimately, related to its polarizability. And rather than having this Coulomb interaction, you would have some dissociation energy or something else, or the density itself would come over here. So, the only final step is that I have to regard my system having a composition of these things of different sizes. So I have to do an integral over r, as well as orientation. So I have to do an integral over E theta. Of course, the integration will go from a through essentially, the size of the system or infinity.

I forgot one other thing, which is that when I'm calculating how many places I can put this, again, I have been calculating things per unit area of a squared. So I would have to divide all of these places where r and L appear by corresponding factors of a. OK?

So, the last step of the calculation is you expand this quantity. It is 1. For small values of the electric field, it is 2 pi r E prime cosine of theta k plus higher order terms. And then, you can do the various integrations. First of all, 1 the integration against 1 will disappear because you are integrating over all values of cosine of theta. Integral of cosine of theta gives you zero. Essentially, it says that if there was

no electric field, there was no reason for there to be an additional net charge on one side or the other. So the first term that will be non-zero is the average of cosine theta squared, which will give you a factor of one-half. And so, what you will get is that E minus E prime times L is-- well, there's going to be a factor of L. The integral of the theta cosine of theta squared-- the integral of cosine of theta squared is going to give you 2 pi, which is the integration times one-half. So this is the integral the theta cosine square theta will give you this. So we did this.

We have two factors of y. y is our expansion parameter. We are at a low density limit. We've calculated things assuming that essentially, I have to look at one value of these diplodes. In principal, I can imagine that there will be multiple dipoles. And you can see that ultimately, therefore, potentially, I have order of y to the fourth that I haven't calculated. OK, so we got rid of the y squared. We have a factor of E prime on the expansion here.

This factor is bothering me a little bit-- let me check. No, that's correct.

OK, so I have the factor of k. I have a factor of 2 pi here that came from the charge. I have another 2 pi here-- so I have 4pi squared. I think I got everything except the integration over a to infinity dr. There is this r dr, which is from the two dimensional integration. There was another r here, and another r here. So this becomes r to the three. From here, I have minus 2 pi k. And then, I have the corresponding factors of a to the power of 2 pi k minus 4. OK.

So you can see that the L's cancel. And what I get is that E-- once I take the E prime to the other side-- becomes E prime 1 plus I have 4pi cubed k y squared-- again, y squared is my small expansion parameter. And then, I have the integral from a to infinity, the r, r to the power of 3 minus 2 pi k, a to the power of 2 pi k minus 4, and then order of y squared, y to the fourth. OK.

So basically, you see that the internal electric field is smaller than the external electric field by this factor, which takes into account the re-orientation of the dipoles in order to screen the electric field. And it is proportional in some sense to the density of these dipoles. And the twist is that the dipoles that we have can have a

range of sizes that we have to integrate [INAUDIBLE]. So typically, you would write E prime to the E over epsilon. And so this is the inverse of your epsilon. And essentially, this is a reduction in everything that has to do with electric interactions because of the screening of other things. I can write it in the following fashion. I can say that there is an effective k-- that's called k effective-- which is different from the original k that I have. It is reduced by a factor of epsilon.

So we were worried when we were doing the nonlinear sigma model that for any [INAUDIBLE], we saw that the parameter k was not getting modified because of the interactions among the spin modes, and that's correct. But really, at high temperatures, it should disappear. We saw that the correlations had to go away from power law form to exponential form. And so, we needed some mechanism for reducing the coupling constant. And what we find here is that this topological defects and their screening provide the right mechanism. So the effective k that I have is going to be reduced from the original k by the inverse of this. Since I'm doing an expansion in y, it is simply minus 4pi cubed ky squared, integral a to infinity dr, r to the 3 minus 2 pi k, a to the 2 pi k minus 4, plus order or y to the fourteenth. OK.

Now actually, in the lecture notes that I have given you, I calculate this formula in an entirely different way. What I do is I assume that I have two topological defects-- so there I sort of maintain the picture of topological defect. And their interaction between them is this logarithmic interaction that has coefficient k. But then, we say that this [INAUDIBLE] interaction is modified because I can create pairs of topological defect, such as this, that will partially screen the interaction. And in the notes, we calculate what the effect of those pairs at lowest order is on their interaction that you have between them. And you find that the effect is to modify the coefficient of the logarithm, which is k, to a reduced k. And that reduced k is given exactly by this form. So the same thing you can get different ways. Yes?

AUDIENCE: What if k is too small--

PROFESSOR: A-ha. Good. Because I framed the entire thing as if I'm doing a preservation theory

for you in y being a small parameter. OK? But now, we see that no matter how small y is, if k is in fact less than 2 over pi-- so this has dimensions of r to the 4 minus 2 pi k. So if k is less than 2 over pi-- which incidentally is something that we saw earlier-- if k is less than that, this integral diverges. So I thought I was controlling my expansion by making y arbitrarily small, but what we see is that no matter how small I make y, if k becomes too small, the perturbation acuity blows up on me. So this is yet another example of a singular perturbation theory, which is what we had encountered when we were doing the Landau-Ginzburg model. We thought that our co-efficient of phi to the fourth u was a small parameter. You are making an expansion naively in powers of u. And then we found an expression in which the coeffecient-- the thing that was multiplying u at the critical point was blowing up on us. And so the perturbation theory inherently became singular, despite your thinking that you had a small parameter. So we are going to use the same trick that we used for the case of the Landau-Ginzburg model-- this is deal with singular perturbations by renormalization group. So what we see is that the origin of the problem is the divergence that we get over here when we try to integrate all the way to infinity or the size of the system.

So what we do instead is we said, OK, let's not integrate all the way. Let's replace the short distance cut-off that we had with something that is larger-- ba-- and rather than integrating all of a to infinity, we integrate only over short distance fluctuations between a and ba. This is our usual [INAUDIBLE]. So what we therefore get is that the k effective is k 1 minus 4pi q ky squared integral from a to ba br r to the 3 minus 2 pi k a to the 2 pi k minus 4.

And then, I have to still deal with 4pi cubed ky squared, integral from ba infinity dr r to the 3 minus 2 pi k, a to the 2 pi k minus 4, plus order of y to the fourth. OK? All right.

So, you can see that the effect of integrating this much is to modify the decoupling to a new value which depends on b, which is just k minus 4pi cubed k squared y squared, a to ba br r to the 3 minus 2 pi k. a to the 2 pi k minus 4. OK? And then, I can rewrite the expression for k effective to be this k tilde. And then, whatever is left,

which is 4pi cubed k squared y squared integral ba to infinity dr r to the 3 minus 2 pi k, a to the 2 pi k minus 4, order of y to the fourth.

You see that k has been shifted through this transformation by an amount that is order of y squared. So at order of y squared in this new expression, I can replace all the k's that are carrying with k tilde and it would still be correct for this order.

Now I compare this expression and the original expression that I had. And I see that they are pretty much the same expression, except that in this one, the cut-off is ba. So I do step two of origin. I define my R prime to be dr so that my new cut-off will be back to a. So then, this whole thing becomes k effective is k tilde minus 4pi cubed k tilde squared y squared. Because of the transformation that I did over here, I will get a factor of b to the 4 minus 2 pi k tilde, integral from a' to infinity-- the r prime-- r prime to the 3 minus 2 pi k tilde, a to the 2 pi k tilde, minus 4 plus order of y to the fourteenth.

So we see that the same effective interaction can be obtained from two theories that have exactly the same cut-off, a, except that in one case, I had k and y. In the new case, I have this tilde or k prime at scale b. And I have to replace where I had y with y with this additional factor. So the two theories are equivalent provided that I say that the new interaction at scale b is the old interaction minus 4pi cubed k squared y squared. This integral is easy to perform. It is just the power law. It is b to the fourth minus 2 pi k minus 1. And then, I have 4 minus 2 pi k order of y to the fourth. And my y prime is y-- from here I see b to the power of 2 minus pi k.

AUDIENCE: [INAUDIBLE]

PROFESSOR: This k squared?

AUDIENCE: Oh, I see. Sorry.

PROFESSOR: All right. So, our theory is described in terms of two parameters-- this y and this k. or let's say, it's inverse-- k inverse, which is more like temperature. And what we will show next time is that these recursion relations, when I draw it here, will give me two types of behavior. One set of behavior that parameterizes the low temperature

dilute limit that corresponds to flows in which y goes through zero. So that when you look at the system at larger and larger lens scales, essentially it becomes less and less depleted of these excitations. So, once you have integrated the very essentially, you don't see any excitations.

And then there's another phase, which as you do this removal of short distance fluctuations. You tend to flow to high temperatures and large densities. And so, that corresponds to this kind of face. Now the beauty of this whole thing is that these recursion relations are exact and allow us to exactly determine the behavior of these space transition in two dimensions. And that's actually one of the other triumphs of renormalization group is to elucidate exactly the critical behavior of this transition, as we will discuss next time.