8.324 Relativistic Quantum Field Theory II

MIT OpenCourseWare Lecture Notes

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Lecture 23

5.2.2: Fixed Points of Renormalization Flow

What is the end point of a renormalization group flow as we take $\Lambda \longrightarrow 0$? This is one of the most important physical questions. Let us consider the possibilities:

1. Some couplings become very strong. This leads to the formation of bound states, and to new infrared degrees of freedom, as in quantum chromodynamics.

2. $\{\lambda_j\}$ has a fixed point. That is, $\beta_i(\{\lambda_j\}) = 0$, and so

$$\Lambda \frac{d\lambda_i}{d\Lambda} = 0, \tag{1}$$

meaning $\lambda_i(\Lambda) = \lambda_i^*$ is a constant, independent of scale. At a fixed point, a theory is scale invariant.

As a trivial example of this second case, we will consider a Gaussian fixed point. Let the Lagrangian be given by

$$\mathscr{L} = \frac{1}{2}\left(\partial\phi\right)^2, \tag{2}$$

describing a free, massless particle. In this case, $S_i = 0$, and all of the couplings satisfy $\lambda_i = 0$, and this is clearly invariant under the flow. In fact, our earlier discussion of RG flow and the notion of relevant and irrelevant couplings are defined implicitly with respect to this fixed point.

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{int}, \tag{3}$$

where $\mathscr{L}_0 = \frac{1}{2}\left(\partial\phi\right)^2$ and $\mathscr{L}_{int} = \sum_i g_i O_i$. The dimensions Δ_i of O_i are found from the canonical dimension of ϕ, which is defined using $\mathscr{L}_0$. For a generic fixed point, $\lambda_i(\Lambda) = \lambda_i^*$ constant, consider a small deviation from the fixed

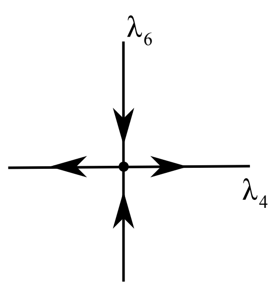


Figure 1: Flow diagram for the dimensionless couplings, λ_4 and λ_6. λ_4 is marginally irrelevant.

point,

$$\delta\lambda_i = \lambda_i - \lambda_i^*. \tag{4}$$

Then,

$$\begin{aligned} \Lambda \frac{d\delta\lambda_i}{d\Lambda} &= \beta_i\left(\{\lambda_j^* + \delta\lambda_j\}\right) \\ &= \sum_j \left.\frac{d\beta_i}{d\lambda_j}\right|_{\lambda_j^*} \delta\lambda_j + \dots \\ &= \sum_j T_{ij}\delta\lambda_j + \dots \end{aligned}$$

with $T_{ij} = \frac{d\beta_i}{d\lambda_j}\Big|_{\lambda_j^*}$. We now consider the left eigenvectors, $e_i^{(\alpha)}$, of T, with eigenvalues t_α, where α indexes the eigenvectors. That is,

$$\sum_i e_i^{(\alpha)} T_{ij} = t_\alpha e_j^{(\alpha)}. \tag{5}$$

Now, we let

$$U_\alpha = \sum_i e_i^{(\alpha)} \delta\lambda_i. \tag{6}$$

Then

$$\Lambda \frac{dU_\alpha}{d\Lambda} = t_\alpha U_\alpha + O(U^2). \tag{7}$$

Disregarding the non-linear terms, we suppose at some Λ_0 that $U_\alpha = U_\alpha^{(0)}$. Then we have that

$$U_\alpha(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^{t_\alpha} U_\alpha^{(0)}. \tag{8}$$

U_α is the scaling variable, and $-t_\alpha = [U_\alpha]$ is the scaling dimension, as defined with respect to the fixed point $\{\lambda^*\}$. We have that

$$-t_\alpha = [U_\alpha] = \begin{cases} > 0 & \text{relevant,} \\ = 0 & \text{marginal,} \\ < 0 & \text{irrelevant.} \end{cases} \tag{9}$$

Now,

$$\mathscr{L} = \mathscr{L}_* + \mathscr{L}_{int}, \tag{10}$$

where

$$\begin{aligned} \mathscr{L}_* &= \sum_i \lambda_i^* O_i, \\ \mathscr{L}_{int} &= \sum_i \delta\lambda_i O_i = \sum_\alpha U_\alpha \tilde{O}_\alpha, \end{aligned}$$

where $\tilde{O}_\alpha$ are the scaling operators with respect to $\mathscr{L}_*$. We have that $\left[\tilde{O}_\alpha\right] = d + t_\alpha$. Now consider the case of U_1, U_2 with $t_1 < 0$ (relevant) and $t_2 > 0$ (irrelevant). Then

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = M \begin{pmatrix} \delta\lambda_1 \\ \delta\lambda_2 \end{pmatrix}. \tag{11}$$

The **critical surface** is the submanifold in the space of couplings spanned by irrelevant couplings near a fixed

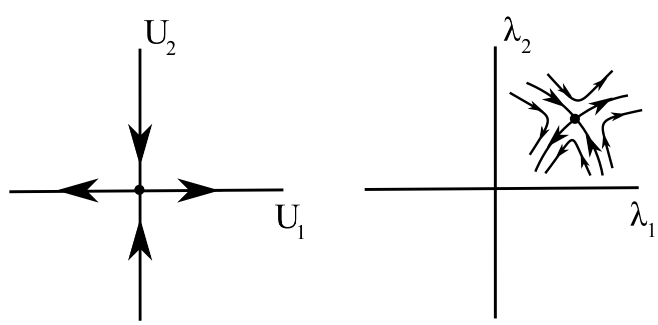


Figure 2: Flow in the U_1, U_2 plane and the λ_1, λ_2 plane about the fixed point λ^*.

point. Any point on the critical surface will flow to the fixed point. A point lying off the critical surface will flow away from it, as shown in figure 3. The critical surface has a finite co-dimension, since the number of relevant and marginal couplings is always finite.

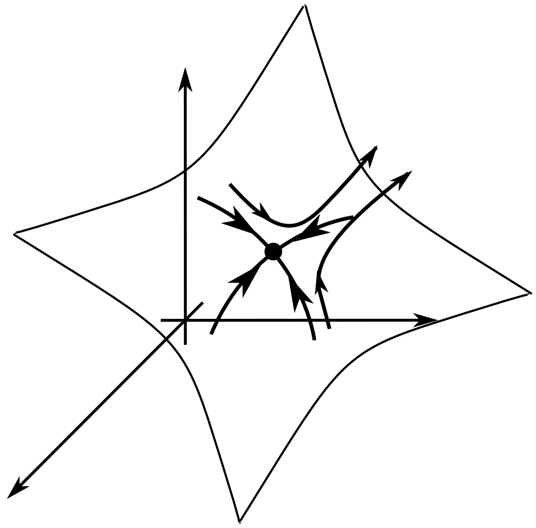


Figure 3: The critical surface submanifold. Couplings on the manifold are irrelevant and flow towards the fixed point. Couplings off the manifold flow away from it.

Let us consider, at the ultraviolet scale, Λ_0, $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{int}$, with $\mathscr{L}_0 = \frac{1}{2}\left(\partial\phi\right)^2$, and $\mathscr{L}_{int} = \sum_i \lambda_i O_i$ giving deviations from the fixed point.

$$O_i = \phi^2,\ \left(\partial\phi\right)^2,\ \phi^4,\ \ldots \tag{12}$$

are scaling operators with respect to $\mathscr{L}_0$. In general near $\{\lambda_i^*\}$, the spectrum of scaling operators $\{O_\alpha\}$ and their conformal dimensions will be very different from those at the Gaussian fixed point. Such non-trivial fixed points are very rare at a finite distance away from the Gaussian fixed point, particularly in $d \geq 3$. For this reason, they provide a very limited theoretical tool. We consider one famous example:

$$\mathscr{L} = \frac{1}{2}\left(\partial\phi\right)^2 + \frac{\lambda}{4!}\phi^4 \tag{13}$$

at Λ_0 in $d = 4$. Then we have

$$\beta_\lambda = c\lambda^2 + \ldots, \tag{14}$$

where c is a positive constant, giving a marginally irrelevant flow back to the Gaussian fixed point. If we now consider $d = 4 - \epsilon$, then

$$[\phi] = \frac{d-2}{2} = 1 - \frac{\epsilon}{2}, \quad [\lambda] = \epsilon, \tag{15}$$

for $\epsilon \ll 1$. Then we have

$$\beta_\lambda = -\epsilon\lambda + c\lambda^2 + \ldots, \tag{16}$$

and so, the fixed point is given by

$$\lambda_* = \frac{\epsilon}{c} \ll 1. \tag{17}$$

This fixed point is infinitesimally close to the Gaussian fixed point and can be studied using perturbation theory. We now conclude our discussion of the Wilsonian approach to renormalization. Our discussion has been more focused on the conceptual picture, rather than the explicity calculation. The Wilsonian approach is conceptually very appealing, but technically very awkward, as the couplings depend on the scale. We want to find how to see it from the conventional approach.

5.3: BETA-FUNCTIONS FROM THE TRADITIONAL APPROACH

We recall the procedure for renormalization, for example in ϕ^3-theory.

1. Firstly, we have the Lagrangian

$$\mathscr{L}(\phi_B, g_B, m_B) = \mathscr{L}(\phi, g, m) + \mathscr{L}_{ct}, \tag{18}$$

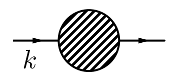
where

$$\mathscr{L} = -\frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}m^2\phi^2 + \frac{g}{6}\mu^{\frac{\epsilon}{2}}\phi^3 \tag{19}$$

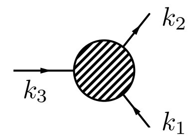
and

$$\mathscr{L}_{ct} = -\frac{1}{2}A\left(\partial\phi\right)^2 - \frac{1}{2}Bm^2\phi^2 + \frac{g}{6}\mu^{\frac{\epsilon}{2}}C\phi^3. \tag{20}$$

2. We impose the following renormalization conditions to determine the counter terms:

$\Pi(k^2) =$  (21)

$$\begin{array}{rcl} \Pi(k^2 & = & -m^2) = 0 \quad \text{(Physical mass condition)} \\ \Pi'(k^2 & = & -m^2) = 0 \quad \text{(Physical field condition)} \end{array}$$

$V(k_1, k_2, k_3) =$  (22)

$$V(0,0,0) = g \quad \text{(Physical coupling condition)} \tag{23}$$

3. All physical observables can be expressed in terms of m^2 and g.

We consider more explicitly the one-loop expression for $\Pi(k^2)$ and $V(k_1, k_2, k_3)$:

$$\Pi(k^2) = -\frac{\alpha}{2}\left[\left(\frac{2}{\epsilon}+1\right)\left(\frac{k^2}{6}+m^2\right) + \int_0^1 dx\, D\log\left(\frac{4\pi\mu^2}{e^\gamma D}\right)\right] - Ak^2 - Bm^2, \tag{24}$$

where $D \equiv x(1-x)k^2 + m^2$, $\alpha \equiv \frac{g^2}{(4\pi)^3}$, and

$$\frac{1}{g}V(k_1, k_2, k_3) = 1 + \frac{\alpha}{2}\left[\frac{2}{\epsilon} + K\right] + C, \tag{25}$$

where $K = K(m^2, k_1, k_2, k_3, \mu)$. The explicit form of this function is not important here. In order to cancel the divergences, we require

$$\begin{array}{rcl} A & = & -\frac{\alpha}{6\epsilon} + a, \\ B & = & -\frac{\alpha}{\epsilon} + b, \\ C & = & -\frac{\alpha}{\epsilon} + c. \end{array}$$

where a, b and c are finite constants determined by the renormalization conditions. Therefore, $\Pi(k^2)$ and $V(k_1, k_2, k_3)$ are independent of the auxillary scale μ and depend only on m^2 and g. But we can actually choose a, b and c arbitrarily, and physical observables will be finite. The previously stated renormalization conditions will generally not be satisfied, and so m^2, ϕ and g are no longer the physical mass, field and couplings. They can be interpreted as parameters of the Lagrangian. The conditions given by

$$\begin{array}{rcl} \Pi(k^2 = -m^2_{phys}) & = & 0, \\ \Pi'(k^2 = -m^2_{phys}) & = & 0, \\ V(0,0,0) & = & g_{phys}, \end{array}$$

and are now solved by general

$$\begin{array}{rcl} m_{phys} & = & m_{phys}(m^2, g), \\ \phi_{phys} & = & \phi_{phys}(m^2, g, \phi), \\ g_{phys} & = & g_{phys}(m^2, g), \end{array}$$

which are all finite relations. For example, if we set $0 = a = b = c$, we find

$$g_{phys} = g\left[1 + \frac{\alpha}{2}K(m^2, \mu; k_i = 0)\right], \tag{26}$$

where

$$K = \int_0^1 dx_1 dx_2 dx_3\, \delta(x_1 + x_2 + x_3 - 1) \log\left(\frac{4\pi\mu^2}{e^\gamma \tilde{D}}\right), \tag{27}$$

with

$$\tilde{D} \equiv m^2 + x_2 x_3 k_1^2 + x_1 x_3 k_2^2 + x_1 x_2 k_3^2, \tag{28}$$

and so

$$g_{phys} = g\left[1 + \frac{\alpha x}{2}\log\left(\frac{4\pi\mu^2}{m^2}\right)\right], \tag{29}$$

where x is a numerical constant. A similar result holds for m^2_{phys}. We note that, in general, the $\mu-$dependence does not disappear in such relations. In practice, we choose μ, and from m_{phys} and g_{phys}, that is, from the measured quantities, we determine g and m and hence all other observables. The physical observables, of course, like m_{phys} and g_{phys} do not depend on μ. So we have $g(\mu)$ and $m^2(\mu)$, and

$$\mathscr{L}(\phi_B, g_B, m_B) = \mathscr{L}(\phi, g, m) + \mathscr{L}_{ct}. \tag{30}$$

Here, the left-hand side is μ-independent. The splitting on the right-hand side requires introducing a scale μ. ϕ, g and m^2 should be $\mu-$dependent, so that the physical observables are $\mu-$independent, but the finite part of the split is arbitrary. A different choice of a,b and c is a different choice of scheme: the one we have been using until now is the on-shell scheme. Different schemes give different $g(\mu)$, $\phi(\mu)$ and $m^2(\mu)$. Again, the physics should not depend on such a choice, corresponding to the different ways of doing the splitting.