1. Rel. QM

~~Basi~~

7.1 Basics of SR

Space-time coordinates $x^\mu \quad \mu = 0, 1, 2, 3$
$= (ct, x, y, z)$

proper time
interval $\quad s^2 = \eta_{\mu\nu} x^\mu x^\nu = c^2t^2 - |\vec{x}|^2$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

Lorentz xforms $\quad x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad$ which preserve s^2

satisfy $\quad \eta_{\mu\sigma} \Lambda^\mu{}_\nu \Lambda^\sigma{}_\tau = \eta_{\nu\tau}$

Group of Λ's : Lorentz group
Includes: (det = +1 proper Lor xf
-1 improper " ")

Rotations $\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \quad R \in SO(3)$

Ex. rot around z $\begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}$

Boosts

Ex boost along x, $\quad \Lambda = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & & \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ & 0 & 1 & \\ & 0 & & 1 \end{pmatrix}$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

Improper: spatial inversion
$$\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$$

4-vectors & tensors

4vector: 4-component objects V^μ
xform under $\Lambda^\mu{}_\nu$ as

$$(V')^\mu = \Lambda^\mu{}_\nu V^\nu$$

Ex. momentum $p^\mu = (E/c, \vec{p})$

$$p^2 = p^\mu p_\mu = E^2/c^2 - |\vec{p}|^2 = m^2c^2$$

$$[\, p_\mu = \eta_{\mu\nu} p^\nu, \quad p^\mu = \eta^{\mu\nu} p_\nu, \quad \eta^{\mu\nu}\eta_{\nu\lambda} = \delta^\mu{}_\lambda \,]$$

tensor: $T'^{\mu\nu} = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau T^{\sigma\tau}$

Ex. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ & 0 & B^3 & -B^2 \\ & & 0 & B^1 \\ & & -D^1 & 0 \end{pmatrix}.$$

4.2 Klein - Gordon eqn

We want a relativistic analog of Schrödinger

Schrödinger:

$$E = \frac{p^2}{2m} + V(x)$$

$$E = i\hbar\frac{\partial}{\partial t} \qquad p^i = -i\hbar\frac{\partial}{\partial x^i} \qquad \left(p_\mu = i\hbar\frac{\partial}{\partial x^\mu}\right)$$

Schröd: $H\psi = \left(\frac{p^2}{2m} + V(x)\right) \cancel{\psi(x,t)}$

$$\left[i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(x)\right)\psi(x,t)\right]$$

Relativistic mass relation

$$E^2 = p^2c^2 + m^2c^4$$

$$\Rightarrow \quad -\hbar^2\frac{\partial^2\psi}{\partial t^2} = -\hbar^2 c^2 \vec{\nabla}^2\psi + m^2c^4\psi$$

$$\Leftrightarrow \quad \left(p^\mu p_\mu - m^2c^2\right)\psi = 0$$

$$p_0 = i\hbar\frac{\partial}{\partial t}$$

$$p_i = -i\hbar\frac{\partial}{\partial x^i}$$

$$\Box = \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}$$

$$\Leftrightarrow \boxed{\left[\cancel{\eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}}\,\Box + \frac{m^2c^2}{\hbar^2}\right]\psi(x^\mu) = 0}$$

Klein-Gordon Eqn.

Klein-Gordon = 2nd order diff eq. (Schrödinger 1st order)

Problems with Klein-Gordon:

plane wave solutions: $\psi(x^\mu) = \frac{1}{\sqrt{V}} e^{-\frac{i}{\hbar} p_\mu x^\mu} = \frac{1}{\sqrt{V}} e^{\frac{i}{\hbar} \underline{p} \cdot \underline{x} - \frac{i}{\hbar} E t}$

$$E^2 = p^2c^2 + m^2c^4$$

for given p has $E = \pm |E|$ solutions

negative E solution appears unphysical
— but needed to form complete basis
In presence of potential no consistent way to drop $-E$ solutions
particles would cascade downward to $E \to -\infty$

prob. current. Recall for Schrödinger

$$(\psi)^*\left[i\hbar \dot{\psi} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \right]$$

$$- \left[-i\hbar \dot{\psi}^* = -\frac{\hbar^2}{2m} \nabla^2 \psi^* + V \psi^* \right] \psi$$

$$\Rightarrow \frac{\partial}{\partial t} \rho_s = -\underline{\nabla} \cdot \underline{j}_s$$

$$\rho_s = |\psi|^2 \qquad \text{prob. density}$$

$$\underline{j}_s = -\frac{i\hbar}{2m} \left(\psi^* \underline{\nabla} \psi - \psi \underline{\nabla} \psi^* \right) \qquad \text{prob. current}$$

Try same for KG:

$$\psi^* \left(\left(\Box + \frac{m^2 c^2}{\hbar^2} \right) \psi = 0 \right]$$

$$- \left[\left(\Box + \frac{m^2 c^2}{\hbar^2} \right) \psi^* = 0 \right] \psi^*$$

$$\psi^* \Box \psi - \psi \Box \psi^* = - \partial^\mu \left[\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^* \right] = 0$$

$\Rightarrow$ 4-current $\quad j_\mu = A \left[\psi^* \partial_\mu \psi - \psi \partial_\mu \psi^* \right)$

(A const)

$$\partial^\mu j_\mu = 0.$$

to agree with NR current $\underline{j_s}$, $A = \frac{i\hbar}{2m}$

$$\Rightarrow \quad \vec{j} = \vec{j}_s$$

$$\rho = \frac{1}{c} j_0 = \frac{i\hbar}{2mc^2} \left[\psi^* \frac{\partial}{\partial t} \psi - \psi \frac{\partial}{\partial t} \psi^* \right]$$

In NR limit, $\frac{\partial}{\partial t} \psi = -i (mc^2) \psi$, so $\Rightarrow \rho_s$.

But ρ is not necessarily positive in general.

Cannot interpret as single-particle equation.

Problems arise apparently from 2nd order nature of eqn.
(ψ, $\dot\psi$ both arb. fns at $t=0$)

Historically KG ignored.

Dirac: found 1st order eq. for massive part.

Actually, $\pm E$ problem appears because relativistically, really need many-particle formalism (QFT)

neg. E states in KG are really antiparticles

Pauli & Weisskopf (1934) showed KG ok for scalar field thy - ρ is charge density.

9.3 Dirac eqn

1928: Dirac [looked for & found]
1st order relativistic 1-particle wave equation.

~~Dirac approach: Consider general~~

Analogy: E & M
- [Maxwell eqns are 1st order in E, B, mix cpts
- (each cpt separately satisfies) $\Box E^i = \Box B^i = 0$ BUT
(w/out sources)

Dirac approach: considered
multi-component function ψ_ℓ

1st order matrix differential eqn

$$i\hbar \frac{\partial \psi_\ell}{\partial t} = -i\hbar c\, (\alpha_i)_\ell{}^n \frac{\partial \psi_n}{\partial x^i} + mc^2 \beta_\ell{}^n \psi_n = 0$$

Calculate $\left(i\hbar \frac{\partial}{\partial t}\right)^2 \psi$,

require each cpt satisfies Klein – Gordon $\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0$

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = \left[mc^2\beta - i\hbar c\, \alpha_j \partial_j\right]\left[mc^2\beta - i\hbar c\,\alpha_k \partial_k\right]\psi$$

$$= \Bigg(m^2c^4 \underbrace{\beta}_{1} - i\hbar mc^3 \underbrace{(\beta\alpha_j + \alpha_j\beta)}_{\cdot} \frac{\partial}{\partial x^j} - \hbar^2c^2 \underbrace{\frac{(\alpha_i\alpha_j + \alpha_j\alpha_i)}{2}}_{\delta_{ij}} \frac{\partial^2}{\partial x^i \partial x^j} \Bigg)\psi$$

To get K.G. eq requires

$\beta^2 = 1$
$\beta\alpha_j + \alpha_j\beta = 0$
$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$

i.e.

$\beta^2 = \alpha_i^2 = 1$
$\{\beta, \alpha_i\} = 0$
$\{\alpha_i, \alpha_j\} = 0 \quad i \neq j$

H hermitian $\Rightarrow$ β, α_j Hermitian

$\beta^2 = \alpha_j^2 = 1 \Rightarrow$ eigenvalues ± 1

$\alpha_j\beta = -\beta\alpha_j \quad \Rightarrow \quad \alpha_j = -\beta\alpha_j\beta$

$$\mathrm{Tr}\,\alpha_j = -\mathrm{Tr}(\beta\alpha_j\beta) = -\mathrm{Tr}(\beta^2\alpha_j)$$
$$= -\mathrm{Tr}(\alpha_j) = 0.$$

Dimension of matrices:

Eigenvalues ± 1 & $Tr = 0 \Rightarrow$ even dim

$N = 2$? Need 4 anticommuting matrices

know pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij}$$

But no 4th matrix $\rightarrow \mathbb{1}$ doesn't anticommute.

So smallest Dim is $N = 4$

Explicit realization

$$\beta = \begin{bmatrix} \mathbb{1}_{2\times2} & \\ & -\mathbb{1}_{2\times2} \end{bmatrix} \qquad \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}$$

$$(= \sigma_3 \otimes \mathbb{1}) \qquad (= \sigma^1 \otimes \sigma_i)$$

~~Standard (Covariant)~~

~~Covariant~~ form of Dirac eqn

Covariant ~~...~~ (mult by β/c)

$$i\hbar \frac{\beta}{c} \frac{\partial \psi}{\partial t} = \left[mc + i\hbar \beta \alpha_j \, \partial_j \right] \psi$$

Define $\gamma^0 = \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$

$$\gamma^i = \beta \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

Dirac: $i\hbar\gamma^0\partial_0\psi = (mc - i\hbar\gamma^i\partial_i)\psi$

$$\Rightarrow \boxed{(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0.}$$

often written $(i\hbar\not{\partial} - mc)\psi = 0$

$$\not{\partial} \equiv \gamma^\mu\partial_\mu$$

Rewrite comm. rel's.

$$\gamma^{0\,2} = \beta^2 = 1$$

$$\gamma^0\gamma^i + \gamma^i\gamma^0 = \beta(\beta\alpha_i) + \beta\alpha_i\beta = 0$$

$$\gamma^i\gamma^j + \gamma^j\gamma^i = (\beta\alpha_i)(\beta\alpha_j) + (\beta\alpha_j)(\beta\alpha_i)$$
$$= -\beta^2(\alpha_i\alpha_j + \alpha_j\alpha_i) = -2\delta_{ij}$$

$$\Rightarrow \underline{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}}$$

Klein - Gordon again

$$i\hbar\gamma^\mu\partial_\mu\psi = mc\psi$$

$$-\hbar^2(\gamma^\mu\partial_\mu)(\gamma^\nu\partial_\nu)\psi = m^2c^2\psi$$
$$= -\hbar^2\tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\mu\partial_\nu\psi$$
$$= -\hbar^2\Box\psi \Rightarrow \left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

~~[illegible]~~

current conservation for Dirac

$$\psi^*\left[i\hbar\dot{\psi} = mc^2\beta\psi - i\hbar c\,\alpha_j\frac{\partial}{\partial x^j}\psi\right]$$

$$-\quad\left[-i\hbar\dot{\psi}^* = mc^2(\beta\psi^*)^T + i\hbar c\left(\alpha_j^*\frac{\partial}{\partial x^j}\psi^*\right)^T\right]\psi$$

$$\frac{\partial}{\partial t}(\psi^*\psi) + \frac{\partial}{\partial x^j}\left[c\,\psi^*\alpha_j\psi\right] = 0$$

pos. def. prob. density.

~~[illegible] prob. current~~

Γ [illegible]rties of γ matrices

Can form $2^4 = 16$ lin indep matrices from products of γ's

$$\begin{array}{ccccc} 1, & \gamma_\mu, & \gamma_\mu\gamma_\nu, & \gamma_\mu\gamma_\nu\gamma_\lambda, & \gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma \\ 1 & 4 & 6 & 4 & 1 \end{array}$$

denote Γ_m, $m = 1, \dots, 16$

any 4×4 mtx can be written as $M = \sum C_m \Gamma_m$.
[HW: lin indep]

Pauli's Fundamental theorem

Given any 2 sets of 4×4 matrices γ_μ, γ_μ' satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2\,\cancel{\delta}\,\eta_{\mu\nu} I$$

$$\{\gamma_\mu', \gamma_\nu'\} = 2\,\eta_{\mu\nu} I,$$

$\exists$ non-singular ~~[illegible]~~ matrix S so that

$$\gamma_\mu' = S \gamma_\mu S^{-1}$$

So, only one Dirac equation for ~~[illegible]~~ 4-component ψs. different representations. Above is "Pauli-Dirac" rep; most useful when kinetic energy low.

[~~[illegible]~~ Weyl rep: HW]

((lecture 25 intro)

Relativistic QM.

So far: discussed

Klein-Gordon eqn $\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0$

Dirac eqn $(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

Pauli - Dirac representation : $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

Poincare group: generated by J_{ij} rotation

J_{0i} boosts

P^μ translation

reps labeled by mass $P^\mu P_\mu = m^2$

spin $W^\mu W_\mu = -m^2 s(s+1)$

$$W_\mu = -\tfrac{1}{2}\,\varepsilon_{\mu\nu\lambda\sigma} J^{\nu\lambda} P^\sigma$$

Dirac equation : describes spin-1/2 particle,

ψ_a is 4-component spinor.

$\Rightarrow$ 23-14

Relativistic particles & group theory

nonrel.
QM: understand reps of SU(2)
rel. QM: understand reps of Lorentz/Poincare group

Homogeneous Lorentz group generated by

rotations J_i
boosts K_i

Lie algebra

(may be sign issue)

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$
$$[J_i, K_j] = i\varepsilon_{ijk} K_k$$
$$[K_i, K_j] = -i\varepsilon_{ijk} J_k$$

unified form: $J_{\mu\nu}$, $\mu \in \{0,1,2,3\}$

$$J_{0i} = K_i$$
$$J_{ij} = \varepsilon_{ijk} J_k$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\left(\eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} + \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\sigma} J_{\mu\rho}\right)$$

[Lorentz algebra]

Want to include translation generators P_μ

$$[P_\mu, P_\nu] = 0$$
$$[P_\mu, J_{\rho\sigma}] = -i\left(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho\right)$$

Poincaré Algebra

Forms algebra of ~~group~~ Poincaré group (boosts, rotations, & translations)

Want to consider [Irreps of Poincaré] (Wigner 1939)

$C_1 = P^\mu P_\mu$ commutes with everything (Casimir of) — like J^2 for SU(2)

$\Rightarrow \quad P^\mu P_\mu = m^2$

Reps distinguished by M^2. Also, sign of P^0 unchanged by Lorentz xform.

~~Classes of reps~~

~~$p^2 = m^2 \geq 0$, $p_0 \geq 0$ [physical~~
~~$p^2 = m^2 \geq 0$, $p_0 \leq 0$~~

Consider reps with $p^2 = m^2 > 0$, $p^0 > 0$. (massive physical states)

Choose eigenstate of P^μ, $P^\mu |\phi\rangle = p^\mu |\phi\rangle$.

"Little group" of p^μ: subgroup of Poincaré leaving p^μ fixed $\Rightarrow$ SU(2) in this case.
[e.g. for $p^\mu = (mc, 0, 0, 0)$]

So irreps characterized by M^2, spin s

Spin: from Pauli - Lubanski pseudovector

$$W_\mu = -\frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} J^{\nu\lambda} P^\sigma$$

Clearly $W_\mu P^\mu = 0$.

Can show $C_2 = W^\mu W_\mu = -m^2 s(s+1)$
is Casimir op.;

C_1 & C_2 determine rep. of Poincaré group.

Particles distinguished by mass & spin.

So group thy + QM ⇒ ~~all ... that is QM ...~~ particles have fixed mass, spin.

+opt Dirac eqn describes spin-1/2 particle
ψ is a 4 component spinor, xforms under $s=1/2$ rep. of Poincaré algebra
doubling of spin dof: related to $\pm E$ issue for K-Gordon
needed for $m>0$.

Lorentz covariance of Dirac eqn

$$\left(i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\psi = 0$$

say $\quad x'^\mu = \Lambda^\mu{}_\nu x^\nu$

must have $\quad \psi'_j = S_j{}^k(\Lambda)\psi_k \quad$ for some $S(\Lambda)$

so that

$$\left(i\hbar \gamma^\mu \frac{\partial}{\partial x'^\mu} - mc\right)\psi' = 0$$

$$\frac{\partial}{\partial x^\nu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial x^{\mu'}}$$

$$S\left(i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)S^{-1}S\psi$$

$$= \left[i\hbar\left(S\gamma^\mu S^{-1}\right)\Lambda^\nu{}_\mu \frac{\partial}{\partial x^{\nu\prime}} - mc\right]\psi'$$

$$\Rightarrow \quad \gamma^\nu = \left(S\gamma^\mu S^{-1}\right)\Lambda^\nu{}_\mu$$

$$\boxed{\Lambda^\mu{}_\nu \gamma^\nu = S\gamma^\mu S^{-1}}$$

determines Lorentz xforms S of ψ_j.

ψ : 4-component Lorentz spinor

S gives 4-component Spinor representation of proper Lorentz group:

$$J_i = \frac{i}{4}\varepsilon_{ijk}\,\gamma_j\gamma_k \qquad \text{(rotations)}$$

$$K_i = \frac{i}{2}\gamma_0\gamma_i \qquad \left(\text{generally, } J_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] \equiv \frac{1}{2}\sigma_{\mu\nu}\right)$$

Obey Lorentz algebra
$$\left[\begin{array}{l} [J, J] = i\varepsilon J \\ [J, K] = i\varepsilon K \\ [K, K] = -i\varepsilon J \end{array}\right]$$

Examples:

a) Rotation around z-axis by θ

$$S_R = e^{i\frac{(-i)}{2}\gamma_1\gamma_2\theta}$$

$$\left\{-\gamma_1\gamma_2 = -\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\right\}$$

$$S = e^{\frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\theta}$$

[so 1st 2 components xform as 2-cpt spinor
2nd " " " " " .]

v) Boost in z direction, param u $\quad$ ($\tanh u = \frac{v}{c}$)

$$S_B = e^{i\frac{\eta}{2}\gamma_0\gamma_3 u}$$

$$\left[\gamma_0\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_3 \\ +\sigma_3 & 0 \end{pmatrix} \right]$$

$$S_B = e^{\frac{1}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} u}$$

boosts mix 2 cpt spinors. [HW]

c) reflection: must have $\Lambda^\mu_\nu \gamma^\nu = S\gamma^\mu S^{-1}$ $\quad$ $\Lambda^\mu_\nu = (1, -1, -1, -1)$

$\Rightarrow S\gamma^0 S^{-1} = \gamma^0$, $S\gamma^i S^{-1} = -\gamma^i$: sat by $S = \gamma^0$ ✓

For rotations, S is unitary $\quad S^+ = S^{-1}$

~~For boosts~~ ex. $S = e^{i\frac{\theta}{2}\begin{pmatrix} \sigma_3 & \\ & \sigma_3 \end{pmatrix}}$

$$[\gamma_0, \gamma_i\gamma_j] = 0 \Rightarrow \gamma^0 S^+ \gamma^0 = S^{-1}$$

For boosts S is not unitary $\quad S^+ = S$

ex. $S = e^{\frac{\beta}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}}$

$$\{\gamma_0, \gamma_0\gamma_i\} = +\gamma_i - \gamma_i = 0$$

$$\gamma^0 S^+ \gamma^0 = \gamma^0 S \gamma^0 = S^{-1}$$

so $\gamma^0 S^+ \gamma^0 = S^{-1}$ for boosts + rotations

~~Plane wave solutions to Dirac~~

Bilinear covariants

ψ transforms as $\quad \psi' = S(\Lambda)\psi \quad$ under Lorentz xforms.

Can form invariant bilinears using

$$\bar{\psi} = \psi^{\dagger}\gamma_0$$

$$\bar{\psi}' = [S\psi]^{\dagger}\gamma_0 = \psi^{\dagger}\gamma_0\gamma_0 S^{\dagger}\gamma_0$$

$$= \bar{\psi} S^{-1}$$

$\Rightarrow \bar{\psi}\psi$ is invariant. ~~[illegible]~~

~~we show~~

define $\quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

other ~~invariant~~ f. bilinear

$\bar{\psi}\psi$	scalar	$\bar{\psi}'\psi' = \bar{\psi}\psi$
$\bar{\psi}\gamma_5\psi$	pseudoscalar	$\bar{\psi}'\gamma_5\psi' = \det(\Lambda)\,\bar{\psi}\gamma_5\psi$ $(\gamma^0\gamma_5\gamma^0 = -\gamma_5)$
$\bar{\psi}\gamma^{\mu}\psi$	vector	
$\bar{\psi}\gamma^{\mu}\gamma_5\psi$	pseudovector	
$\bar{\psi}\sigma^{\mu\nu}\psi$	tensor	

Solutions to Dirac for free particle

Go to rest frame $p^i = 0$

$$(i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0$$

$$\Downarrow$$

$$\frac{i\hbar}{c}\gamma^0 \frac{\partial}{\partial t}\psi = mc\psi$$

$$i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\dot{\psi} = \frac{mc^2}{\hbar}\psi$$

	H	S_z
$\psi^{(1)}(t) = e^{-i\frac{(mc^2)}{\hbar}t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	mc^2	$+1/2$
$\psi^{(2)}(t) = e^{-i\frac{(mc^2)}{\hbar}t}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$	mc^2	$-1/2$
$\psi^{(3)}(t) = e^{-i\frac{(-mc^2)}{\hbar}t}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$	$-mc^2$	$+1/2$
$\psi^{(4)}(t) = e^{-i\frac{(-mc^2)}{\hbar}t}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$	$-mc^2$	$-1/2$

General plane wave solution of Dirac

~~More generally,~~ given $E,\ p^i$ $\qquad \psi = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

Dirac : $\left(i\hbar \gamma^0 \partial_0 + i\hbar \gamma^i \partial_i - mc \right) \psi = 0$

$\qquad\qquad \gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

$$\Rightarrow \frac{1}{c} E\, u_A - (p\cdot\sigma)\, u_B - mc\, u_A = 0$$

$$u_A = \frac{c(\sigma\cdot p)}{E - mc^2}\, u_B$$

Similarly $\qquad u_B = \dfrac{c(\sigma\cdot p)}{E + mc^2}\, u_A \qquad \left[c^2(\sigma\cdot p)^2 = p^2c^2 \right]$ ✓

$$\Rightarrow E^2 = m^2c^4 + p^2c^2$$

$$u^{(1)}(p) = \begin{pmatrix} 1 \\ 0 \\ p_3c/E+mc^2 \\ (p_1+ip_2)c/E+mc^2 \end{pmatrix}$$

$$u^{(2)}(p) = \begin{pmatrix} 0 \\ 1 \\ (p_1-ip_2)c/E+mc^2 \\ -p_3c/E+mc^2 \end{pmatrix}$$

$$u^{(3)}(p) = \begin{pmatrix} -p_3c/|E|+mc^2 \\ -(p_1+ip_2)c/|E|+mc^2 \\ 1 \\ 0 \end{pmatrix} \qquad u^{(4)} = \begin{pmatrix} -(p_1-ip_2)c/|E|+mc^2 \\ p_3c/(|E|+mc^2) \\ 0 \\ 1 \end{pmatrix}$$

(can also get from boosting rest sol'ns (hw.))

unnormalized solution

~~pos. E solutions~~

$$\psi = N u^{(1 \text{ or } 2)} e^{i \underline{p} \cdot \underline{x}/\hbar - i E t/\hbar}$$

$$= N u^{(3 \text{ or } 4)} e^{i \underline{p} \cdot \underline{x}/\hbar + i |E| t/\hbar}$$

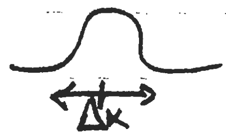
norm. through $u^{(r)+}(p) \; u^{(r)}(p) = \dfrac{|E|}{mc^2}$

$$\Rightarrow \quad N = \sqrt{\frac{mc^2}{|E| V}} \; .$$

Can we ignore neg. E states?

$$\psi(x,0) \sim \begin{pmatrix} \phi(x) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$|\phi|^2$ localized in region



compute coefficients $C_{p,i}$ $i = 1, \ldots 4$

$$\frac{C_{p,3}}{C_{p,1}} \sim -\frac{p_3 c}{|E| + mc^2} \qquad \frac{C_{p,4}}{C_{p,2}} \sim -\frac{(p_1 - ip_2)c}{|E| + mc^2}$$

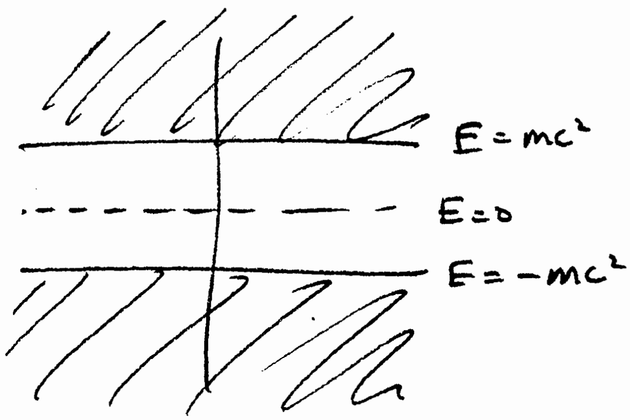
can't ignore when $\Delta p \sim mc$

but this happens when $\Delta x \lesssim \frac{\hbar}{mc}$

so localized states have significant $-E$ components...

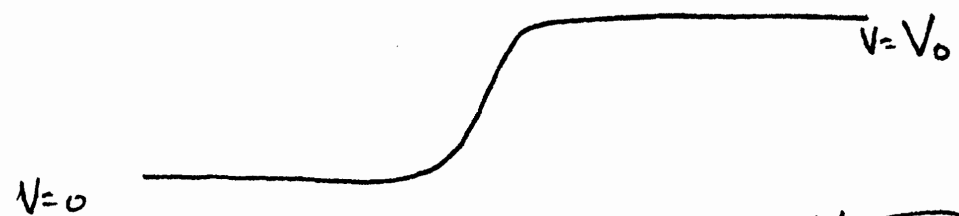
~~Klein paradox~~

~~Relativistic~~
Energies of states



$$\left(E^2 = m^2c^4 + p^2c^2 \right)$$

Klein paradox : consider potential in 1D



~~say $V_0 = E + mc^2$~~

on left, incoming wave

$$p^2c^2 = (E + mc^2)(E - mc^2) > 0,$$

oscillating solution (plane incoming wave)

on right, $p^2c^2 = (E - V + mc^2)(E - V - mc^2)$

when $mc^2 > E - V_0 > -mc^2$,

$p^2c^2 < 0$, so have damped solution.
(usual reflection story)

$E - V_0 \cancel{>} mc^2$, situation changes.

now $p^2c^2 > 0$, oscillating solution on RHS
(neg. E solution)

Find reflection prob > 1!

single-particle interpretation breaks down.

next: Coulomb, hole theory

Dirac eqn in free space

$$(i\hbar \gamma^\mu \partial_\mu - mc)\,\psi = 0$$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,, \quad \psi \text{ 4-cpt spinor.}$$

Found 4 solutions $u^{(i)}(p)$ with fixed $\underline{p}$, 2 have $E>0$, 2 $E<0$.

Interpret negative E solutions as positrons with energy $-E$.

~~For this lecture: $\hbar = c = 1$~~

Angular momentum & the Dirac eqn

$$\begin{pmatrix}\sigma^i & \\ & \sigma^i\end{pmatrix}, \begin{pmatrix}0 & \sigma^l \\ -\sigma^l & 0\end{pmatrix} = i2\varepsilon^{ijk}\begin{pmatrix} & \sigma^k \\ -\sigma^k & \end{pmatrix}$$

~~Conserv. of orb. AM + spin~~

Angular momentum & the Dirac eqn

$$L = \underline{r} \times \underline{p}$$

$$\left[\frac{dL}{dt} = -\frac{i}{\hbar}[L, \gamma^0\gamma^i p_i c + mc^2\gamma^0] \right]$$

$$\frac{dL^i}{dt} = -\frac{i}{\hbar}\left[\varepsilon^{ijk} r_j p_k, \gamma^0\gamma^l p_l c + mc^2\gamma^0\right]$$

$$= c\varepsilon^{ijk}\gamma^0\gamma^j p_k \neq 0$$

$$S^i = \begin{pmatrix}\sigma^i & \\ & \sigma^i\end{pmatrix}\frac{\hbar}{2} \equiv \Sigma^i \frac{\hbar}{2}$$

$$\begin{pmatrix}\sigma^i & \\ & \sigma^i\end{pmatrix}\begin{pmatrix} & \sigma^k \\ -\sigma^k & \end{pmatrix}$$

$$\cancel{i\varepsilon^{ijk}\gamma_j\gamma_k \frac{\hbar}{4}}$$

$$\frac{dS^i}{dt} = -\frac{ic}{2}\left[\Sigma^i, \alpha^j p_j + mc\gamma^0\right]$$

$$\left[\begin{pmatrix}\sigma^i & 0 \\ 0 & \sigma^i\end{pmatrix}, \begin{pmatrix}0 & \sigma^l \\ \sigma^l & 0\end{pmatrix}\right] = 2i\varepsilon^{ijk}\begin{pmatrix}0 & \sigma^k \\ \sigma^k & 0\end{pmatrix}$$

$$\Rightarrow \frac{dS^i}{dt} = c\varepsilon^{ijk} p_j \alpha^k = c\varepsilon^{ijk} p_j \gamma^0\gamma^k$$

$$\Rightarrow \frac{d(L+S)}{dt} = 0 \quad \checkmark$$

Essential feature of Dirac: space & spin DOF combined

Dirac eqn in EM field

Replace

$$E \to E - e\phi$$

$$\underline{p} \to \underline{p} - \frac{e\underline{A}}{c}$$

Equivalent to

$$\partial_\mu \to D_\mu = \partial_\mu - \frac{ie}{\hbar c} A_\mu$$

(gauge-covariant derivative)

Set $\hbar = c = 1$ for convenience

Dirac:

$$i\gamma^\mu (\partial_\mu - ie A_\mu)\psi - m\psi = 0$$

$$(i\not{D} - m)\psi = 0$$

$$-(i\not{D} + m)(i\not{D} - m)\psi = 0$$

$$\Rightarrow 0 = (\not{D}\not{D} + m^2)\psi$$

$$= \left(\tfrac{1}{4}\{\gamma^\mu, \gamma^\nu\}\{D_\mu, D_\nu\} + \tfrac{1}{4}[\gamma^\mu, \gamma^\nu][D_\mu, D_\nu] + m^2\right)\psi$$

$$= \left(D_\mu D^\mu - \tfrac{i}{2}\sigma^{\mu\nu}(-ie F_{\mu\nu}) + m^2\right)\psi$$

$$= \Big(\underbrace{m^2 + D^\mu D_\mu}_{\text{Klein-Gordon in ext field}} - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\Big)\psi$$

Extra terms from E & B fields

$$\left[m^2 + D^\mu D_\mu - e(\underline{\Sigma} \cdot \underline{B}) + ie(\underline{\alpha} \cdot \underline{E})\right]\psi = 0$$

Nonrelativistic limit $\quad \dfrac{(p - ieA)^2}{2m} \qquad -\dfrac{e\hbar}{2mc}\, \underline{\sigma} \cdot \underline{B}$

$\Downarrow$

$g = 2$ for electron, predicted by Dirac thy.

Relativistic Hydrogen Atom

$$H = \gamma^0 \gamma^i p_i + \gamma^0 mc^2 + V(r)$$

Constant of motion: $\quad \underline{J} = \underline{L} + \underline{S}$

Define

$$K = \gamma^0 \underline{\Sigma} \cdot \underline{J} - \gamma^0 \frac{\hbar}{2}$$

Can show $\quad [H, K] = [\underline{J}, K] = 0.$

Eigenstates of H classified by ev's

$$\begin{aligned} H &: E \\ J^2 &: \hbar^2 j(j+1) \\ J_z &: \hbar m \\ K &: -\kappa\hbar \end{aligned}$$

$$\boxed{K = \gamma^0\left(\underline{\Sigma}\cdot\left(\underline{L} + \tfrac{\hbar}{2}\underline{\Sigma}\right) - \tfrac{\hbar}{2}\right) = \gamma^0\left(\underline{\Sigma}\cdot\underline{L} + \hbar\right)}$$

since $[\gamma^0, \underline{\Sigma}] = 0$.

$$\Rightarrow K^2 = \left[(\underline{\Sigma}\cdot\underline{L} + \hbar)^2\right]$$

$$= L^2 + \underbrace{\hbar\,\underline{\Sigma}\cdot\underline{L}}_{\text{(careful)}} + \hbar^2$$

But $J^2 = L^2 + \hbar\,\underline{\Sigma}\cdot\underline{L} + \frac{3}{4}\hbar^2$

$$\Rightarrow K^2 = J^2 + \frac{1}{4}\hbar^2$$

$$\left[K^2 = j(j+1) + \frac{1}{4} = \left(j+\frac{1}{2}\right)^2\right]$$

$$\Rightarrow K = \pm\left(j+\frac{1}{2}\right)$$

+/− : spin parallel / antiparallel to total AM. in nonrel. limit.

$j = 1/2$: $K = \pm 1$

$j = 3/2$: $K = \pm 2$

$\vdots$

$$K = \gamma^0(\underline{\Sigma}\cdot\underline{L} + 1) = \begin{pmatrix} \underline{\sigma}\cdot\underline{L} + 1 & 0 \\ 0 & -(\underline{\sigma}\cdot\underline{L} + 1) \end{pmatrix}$$

Looking for eigenfunctions $\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$
of K, J^2, J_z

eval	K	J^2	J_z
ψ_A	$-K\hbar$	$j(j+1)\hbar^2$	$m\hbar$
ψ_B	$K\hbar$	"	"

but $L^2 = J^2 - \hbar \underline{\sigma} \cdot \underline{L} - \frac{3}{4}\hbar^2$

so $\psi_{A,B}$ are also efuns of L^2, with

$$-K = j(j+1) - l_A(l_A+1) + \frac{1}{4}$$

$$K = j(j+1) - l_B(l_B+1) + \frac{1}{4}$$

$K = j + \frac{1}{2} \Rightarrow \quad l_A = j + \frac{1}{2} \qquad l_B = j - \frac{1}{2}$

$K = -j - \frac{1}{2} \Rightarrow \quad l_A = j - \frac{1}{2} \qquad l_B = j + \frac{1}{2}$

ψ_A, ψ_B not eigenfunctions of L_3 or Σ_3.

can write
$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} g(r)\, \mathcal{Y}_{j\,l_A}^{j_3} \\ i f(r)\, \mathcal{Y}_{j\,l_B}^{j_3} \end{pmatrix}$$

$\mathcal{Y}_{j\ell}^{j_3}$ are normalized spin-angular functions

Explicitly, for ~~$j=\ell$~~

for $j=\ell+1/2$
$$\mathcal{Y}_{j\ell}^{j_3} = \sqrt{\frac{\ell+j_3+\frac{1}{2}}{2\ell+1}}\, Y_{\ell\, j_3-1/2} \begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{\ell-j_3+\frac{1}{2}}{2\ell+1}}\, Y_{\ell\, j_3+1/2} \begin{pmatrix}0\\1\end{pmatrix}$$

for $j=\ell-1/2$
$$\mathcal{Y}_{j\ell}^{j_3} = -\sqrt{\frac{\ell-j_3+\frac{1}{2}}{2\ell+1}}\, Y_{\ell\, j_3-1/2} \begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{\ell+j_3+\frac{1}{2}}{2\ell+1}}\, Y_{\ell\, j_3+1/2} \begin{pmatrix}0\\1\end{pmatrix}$$

To solve Dirac in potential $V(r)$, just need eqs for $f(r)$, $g(r)$

Write Dirac

$$c(\underline{\sigma}\cdot\underline{p})\,\psi_B = (E - V(r) - mc^2)\,\psi_A$$

$$c(\underline{\sigma}\cdot\underline{p})\,\psi_A = (E - V(r) + mc^2)\,\psi_B$$

Now, $\left[\frac{\underline{\sigma}\cdot\underline{x}}{r}\right]\mathcal{Y}_{j\,l_A}^{j_3}$ (pseudoscalar)

gives efunction of J^2, J_3, L^2 with same j, j_3, but opp. orbital parity

so $= \mathcal{Y}_{j\,l_B}^{j_3}$ up to sign

$$(\underline{\sigma}\cdot\underline{p})\,\psi_B = -\hbar\frac{df}{dr}\mathcal{Y}_{j\,l_A}^{j_3} - \frac{(1-K)\hbar}{r}f\,\mathcal{Y}_{j\,l_A}^{j_3}$$

sim. for ψ_A

writing $F(r) = r f(r)$, $G(r) = r g(r)$

$$\Rightarrow \boxed{\begin{aligned} \hbar c\left(\frac{dF}{dr} - \frac{K}{r}F\right) &= -(E - V - mc^2)\,G \\ \& \quad \hbar c\left(\frac{dG}{dr} + \frac{K}{r}G\right) &= (E - V + mc^2)\,F \end{aligned}}$$

set $\hbar = c = 1$, $V = -\frac{Ze^2}{r} = -\frac{Z\alpha}{r}$

Write

$$F = e^{-\rho}\rho^s\sum_{m=0} a_m\rho^m$$

$$G = e^{-\rho}\rho^s\sum_{m=0} b_m\rho^m$$

power series must terminate at n' - use large r, small r behavior to solve

energy levels

$$E = \frac{mc^2}{\sqrt{1 + \frac{Z^2\alpha^2}{\left(n' + \sqrt{(j+1/2)^2 - Z^2\alpha^2}\right)^2}}}$$

usual $\quad n = n' + |K| = n' + j + 1/2$

Expansion

$$E = mc^2\left[1 - \frac{1}{2}\underbrace{\frac{(Z\alpha)^2}{n^2}}_{\text{Balmer}} - \frac{1}{2}\frac{(Z\alpha)^4}{n^3}\underbrace{\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right)}_{\text{fine structure}} + \dots\right)$$

to go to higher order in α,
need 2nd quantization