Lecture 4 (Sep. 18, 2017)

4.1 Measurement

4.1.1 Spin-$\frac{1}{2}$ Systems

Last time, we said that a general state in a spin-$\frac{1}{2}$ system can be written as

$$|\psi\rangle = c_+|+\rangle + c_-|-\rangle\,, \tag{4.1}$$

where $|+\rangle, |-\rangle \in \mathcal{H}$ are eigenstates of S^z and $c_+, c_- \in \mathbb{C}$. We noted that only the relative phase between c_+ and c_- was physically relevant.

Assume that $|\psi\rangle$ is normalized, so that $\langle\psi|\psi\rangle = 1$, which implies that

$$|c_+|^2 + |c_-|^2 = 1\,. \tag{4.2}$$

Now consider what happens when we measure S_z. We find

$$\begin{aligned}\text{Prob}\left(S^z = +\frac{\hbar}{2}\right) &= |\langle+|\psi\rangle|^2 = |c_+|^2\,,\\ \text{Prob}\left(S^z = -\frac{\hbar}{2}\right) &= |\langle-|\psi\rangle|^2 = |c_-|^2\,.\end{aligned} \tag{4.3}$$

We now want to consider a measurement of spin along an arbitrary axis $\hat{\boldsymbol{n}}$. In general, we have

$$\text{Prob}\left(\vec{S}\cdot\hat{\boldsymbol{n}} = \pm\frac{\hbar}{2}\right) = \left|\left\langle \vec{S}\cdot\hat{\boldsymbol{n}} = \pm\tfrac{\hbar}{2}\middle|\psi\right\rangle\right|^2, \tag{4.4}$$

where

$$\left|\vec{S}\cdot\hat{\boldsymbol{n}} = \pm\tfrac{\hbar}{2}\right\rangle \tag{4.5}$$

is the eigenket of the operator $\vec{S}\cdot\hat{\boldsymbol{n}}$ with the eigenvalue $\pm\frac{\hbar}{2}$. This uses the fact that the eigenvalues of the spin projection along any axis are always $\pm\frac{\hbar}{2}$. We can deduce this by noting that we could have chosen any axis originally as the quantization axis, but you should also try to prove it by expressing a general $\vec{S}\cdot\hat{\boldsymbol{n}}$ as a matrix and computing its eigenvalues.

For example, the eigenkets of S^x are

$$\left|S^x = +\tfrac{\hbar}{2}\right\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)\,, \tag{4.6}$$

with eigenvalues $\pm\frac{\hbar}{2}$. We can prove this using the fact that S^x corresponds to the matrix

$$S^x \leftrightarrow \frac{\hbar\sigma^x}{2} = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{4.7}$$

We can then compute

$$\begin{aligned}\text{Prob}\left(S^x = +\frac{\hbar}{2}\right) &= \left|\left\langle S^x = +\tfrac{\hbar}{2}\middle|\psi\right\rangle\right|^2\\ &= \left|\frac{1}{\sqrt{2}}(\langle+| + \langle-|)(c_+|+\rangle + c_-|-\rangle)\right|^2\\ &= \left|\frac{1}{\sqrt{2}}(c_+ + c_-)\right|^2\\ &= \frac{1}{2}|c_+ + c_-|^2\,.\end{aligned} \tag{4.8}$$

4.1.2 The Stern–Gerloch Filter Revisited

Recall the experiment discussed in the first lecture, in which we feed a beam of spin-$\frac{1}{2}$ atoms through a region of spatially-varying magnetic field, resulting in the beam splitting into two beams, one of which has $S^z = +\frac{\hbar}{2}$ and the other of which has $S^z = -\frac{\hbar}{2}$. We discussed several thought experiments using such filters oriented along various axes:

1. First, we discussed feeding the beam through a z-axis filter, and then taking the $S^z = +\frac{\hbar}{2}$ output and feeding it through yet another z-axis filter. The result is that all of the atoms come out of the filter with $S^z = +\frac{\hbar}{2}$. This corresponds to the collapse of the wavefunction; the wavefunction has collapsed after the first measurement, and so the second measurement returns the same result without further altering the wavefunction.

2. Next, we considered feeding the beam through a z-axis filter, and then taking the $S^z = +\frac{\hbar}{2}$ output and feeding it through an x-axis filter. The result is that 50% of the atoms come out with each of $S^x = \pm\frac{\hbar}{2}$. We can see this from the calculation in Eq. (4.8), with $c_+ = 1, c_- = 0$.

3. Finally, we considered feeding the beam through a z-axis filter, feeding the $S^z = +\frac{\hbar}{2}$ output through an x-axis filter, and then feeding the $S^x = +\frac{\hbar}{2}$ output through another z-axis filter. The result is again that 50% of the atoms come out with each of $S^z = \pm\frac{\hbar}{2}$. In this context, we reached the conclusion that we cannot simultaneously measure S^z and S^x. These are incompatible observables.

4.2 Compatible and Incompatible Observables

We have seen that S^z and S^x cannot be simultaneously sharp (just like x and p). When can two observables be simultaneously sharp? Let A, B be two Hermitian operators corresponding to two observables. If we measure A, then the wavefunction collapses to one of the eigenstates $|a\rangle$ of A with eigenvalue a. We next want to measure B and see what happens. Suppose that $|a\rangle$ is also an eigenstate of B with eigenvalue b; in this case, measuring B will return the corresponding eigenvalue b. If we remeasure A, then we will once again find the result a. Any measurements of these two operators will always return sharp values, without changing the state after the first measurement.

Thus, the operators A, B can be simultaneously measured to have sharp values if and only if *all* eigenstates of A are also eigenstates of B. Two such observables are called *compatible*, and two observables not satisfying this relation are called *incompatible*. We can make this statement more concise. If the eigenstates of A are exactly those of B, then A and B are simultaneously diagonalizable, which occurs if and only if $[A, B] = 0$ (we proved this in the last lecture). This is the concise condition for compatibility: two observables are compatible if and only if the corresponding operators commute.

As an example, for the spin-$\frac{1}{2}$ system, the operators $S^2 = S_x^2 + S_y^2 + S_z^2$ and S^z (or the spin along any given axis) are compatible operators, while S^z and S^x are incompatible. Note that

$$[S^z, S^x] = i\hbar S^y \neq 0\,. \tag{4.9}$$

Given a Hilbert space, we can ask what is the maximum number of mutually compatible observables we can find. A *complete set of commuting observables* is a set of observables $\{A, B, C, \ldots\}$ such that all pairwise commutators vanish,

$$[A, B] = [A, C] = [B, C] = \cdots = 0\,, \tag{4.10}$$

and such that for any $a, b, c, \ldots$, there is at most one solution to the eigenvalue equation

$$\begin{aligned} A|\alpha\rangle &= a|\alpha\rangle\,, \\ B|\alpha\rangle &= b|\alpha\rangle\,, \\ C|\alpha\rangle &= c|\alpha\rangle\,, \\ &\vdots \end{aligned} \tag{4.11}$$

4.3 The Generalized Uncertainty Relation

Consider two observables A and B. If we measure A, we want to discuss the variance of the probability distribution of possible outcomes. We do this in the standard way: we define the variance of A as

$$\begin{aligned} \Delta A^2 &= \left\langle (A - \langle A\rangle)^2 \right\rangle \\ &= \left\langle A^2 \right\rangle - \langle A\rangle^2 \\ &= \left\langle \psi \middle| A^2 \middle| \psi \right\rangle - (\langle \psi | A | \psi\rangle)^2\,. \end{aligned} \tag{4.12}$$

If $|\psi\rangle = |a\rangle$ is an eigenket of A with eigenvalue a, then

$$\begin{aligned} \left\langle \psi \middle| A^2 \middle| \psi \right\rangle &= a^2\,, \\ \langle \psi | A | \psi \rangle &= a\,, \end{aligned} \tag{4.13}$$

so $\Delta A^2 = 0$. In general, however, the variance will not vanish.

We further define the standard deviation $\Delta A = \sqrt{\Delta A^2}$. The uncertainty relation states that

$$\Delta A \Delta B \geq \frac{1}{2}|\langle [A, B]\rangle|\,. \tag{4.14}$$

(Note that this is not the strongest possible form of the uncertainty relation, but it is the most commonly seen form.)

Proof. We will make use of the Cauchy–Schwarz inequality: for all $|a\rangle, |b\rangle \in \mathcal{H}$,

$$|\langle a|b\rangle|^2 \leq \langle a|a\rangle\langle b|b\rangle\,. \tag{4.15}$$

For a general $|\psi\rangle$, define the operators

$$\begin{aligned} \delta A &= A - \langle A\rangle\,, \\ \delta B &= B - |B\rangle\,, \end{aligned} \tag{4.16}$$

and the states

$$\begin{aligned} |a\rangle &= \delta A|\psi\rangle\,, \\ |b\rangle &= \delta B|\psi\rangle\,. \end{aligned} \tag{4.17}$$

Applying the Cauchy–Schwarz inequality, we then have

$$|\langle \psi|\delta A \delta B|\psi\rangle|^2 \leq \left\langle \psi \middle| (\delta A)^2 \middle| \psi \right\rangle \left\langle \psi \middle| (\delta B)^2 \middle| \psi \right\rangle\,. \tag{4.18}$$

Note that δA and δB are Hermitian, but there is no guarantee that their product is Hermitian. However, we know that we can write

$$\delta A \delta B = \frac{\delta A \delta B + \delta B \delta A}{2} + \frac{\delta A \delta B - \delta B \delta A}{2}\,. \tag{4.19}$$

The first term on the right-hand side is Hermitian, while the second is anti-Hermitian. We can write this more concisely as

$$\delta A\delta B = \frac{1}{2}\{A, B\} + \frac{1}{2}[A, B]\,. \tag{4.20}$$

Thus,

$$\langle\delta A\delta B\rangle = \frac{1}{2}\langle\{\delta A, \delta B\}\rangle + \frac{1}{2}\langle[\delta A, \delta B]\rangle\,. \tag{4.21}$$

The first term on the right-hand side must be real, as the eigenvalues of any Hermitian operator are real, and the second term on the right-hand side must be imaginary, because any anti-Hermitian operator can be written as i times a Hermitian operator. Thus, the squared modulus of this quantity will simply be the sum of the squared moduli of each of these terms. We then have

$$|\langle\delta A\delta B\rangle|^2 = \frac{1}{4}\big(|\langle\{\delta A, \delta B\}\rangle|^2 + |\langle[\delta A, \delta B]\rangle|^2\big)\,. \tag{4.22}$$

This statement leads to the strongest form of the inequality, but also implies the weaker statement

$$|\langle\delta A\delta B\rangle|^2 \geq \frac{1}{4}|\langle[\delta A, \delta B]\rangle|^2\,. \tag{4.23}$$

Now, note that

$$[\delta A, \delta B] = [A - \langle A\rangle, B - \langle B\rangle] = [A, B]\,, \tag{4.24}$$

so we have

$$|\langle\delta A\delta B\rangle|^2 \geq \frac{1}{4}|\langle[A, B]\rangle|^2\,. \tag{4.25}$$

Returning to Eq. (4.18), we then see that

$$\Delta A^2\Delta B^2 \geq |\langle\delta A\delta B\rangle|^2 \geq \frac{1}{4}|\langle[A, B]\rangle|^2\,, \tag{4.26}$$

which completes the proof. □

4.4 Position and Momentum

We will now move on to study observables with continuous eigenvalues. In order to do so, we need to work with infinite-dimensional Hilbert spaces. Consider any Hermitian operator ξ with a continuous spectrum

$$\xi\big|\xi'\big\rangle = \xi'\big|\xi'\big\rangle\,, \quad \xi' \in \mathbb{R}\,. \tag{4.27}$$

How should we regard the overlap of two states, $\langle\xi'|\xi''\rangle$? For a discrete spectrum, we know that the overlap of two distinct (normalized) eigenkets of a Hermitian operator are orthonormal. We want to appropriately generalize this statement. For a discrete spectrum, the orthonormality condition is stated as

$$\big\langle a\big|a'\big\rangle = \delta_{aa'}\,. \tag{4.28}$$

In the continuous case, we generalize this to

$$\big\langle\xi'\big|\xi''\big\rangle = \delta\big(\xi' - \xi''\big)\,, \tag{4.29}$$

where the Dirac delta "function" $\delta(x)$ is defined to be the object satisfying

$$\int_{-\infty}^{\infty} \mathrm{d}x\ \delta(x) = 1 \tag{4.30}$$

with $\delta(x) = 0$ for all $x \neq 0$. This has the property

$$\int_{-\infty}^{\infty} \mathrm{d}x \; \delta(x) f(x) = f(0) \,. \tag{4.31}$$

We can think of this as the limit of a sequence of increasingly peaked functions centered at $x = 0$, each with total area 1.

We will also generalize the complete relation

$$\mathbb{1} = \sum_a |a\rangle\langle a| \tag{4.32}$$

that is familiar from the discrete spectrum case. For continuous spectra, the equivalent statement is

$$\mathbb{1} = \int_{-\infty}^{\infty} \mathrm{d}\xi' \; \big|\xi'\big\rangle\big\langle\xi'\big| \,. \tag{4.33}$$

This is called *resolution of the identity.* For an arbitrary state $|\psi\rangle$, we then have

$$|\psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}\xi' \; \big|\xi'\big\rangle\big\langle\xi'\big|\psi\big\rangle \,. \tag{4.34}$$

We can express an arbitrary inner product as

$$\big\langle\psi'\big|\psi\big\rangle = \int_{-\infty}^{\infty} \mathrm{d}\xi' \; \big\langle\psi'\big|\xi'\big\rangle\big\langle\xi'\big|\psi\big\rangle \,. \tag{4.35}$$