

Electromagnetic waves in dispersive media.

Reading: Schwinger, Chap. 5, 41, 42 (or Jackson, Chap. 7)

1. Linear and circular polarization.

Consider monochromatic plane waves with frequency ω propagating along the z axis in a dielectric characterized by permeabilities ϵ, μ .

a) Construct all linearly polarized wave and circular polarized wave solutions.

b) Consider two linearly polarized waves with $\mathbf{E} \parallel \hat{\mathbf{x}}$ and $\mathbf{E} \parallel \hat{\mathbf{y}}$, respectively. Under what conditions their superposition is characterized by (i) linear (ii) circular polarization?

2. Fresnel's formulas.

A monochromatic wave is incident on a flat surface of a material with dielectric constant ϵ at an angle θ to the normal.

a) Combine the transmitted and reflected wave dispersion relations $\omega(k)$ with the boundary condition at the surface to obtain Snell's refraction law.

b) Consider linearly polarized waves with polarization normal to the plane of incidence. Find the reflected and transmitted wave amplitudes $E_{t,r}$ as a function of the incidence angle θ . Plot schematically the reflection coefficient $R = |E_r/E_0|^2$ dependence on θ separately for the cases $\epsilon > 1$ and $\epsilon < 1$. Show that in the latter case there is a range of angles in which the reflection is total.

c) Now, consider linearly polarized waves with polarization in the plane of incidence. Find the amplitudes of the reflected and transmitted wave as a function of the incidence angle θ . Show that the reflection coefficient vanishes at a certain value of θ , called Brewster's angle. (Thus a nonpolarized radiation reflected at this angle becomes fully polarized.)

3. EM waves in plasma.

Consider monochromatic waves in plasma, described by the model

$$\epsilon(\omega) = 1 - \omega_p^2/\omega^2, \quad \omega_p^2 = 4\pi n e^2/m \quad (1)$$

a) Find the dispersion relation $\omega(k)$. Show that for ω below the cutoff, $\omega < \omega_p$, waves cannot propagate in plasma.

b) A stylized model of ionosphere is a medium described by the dielectric constant (1). Consider the earth with such a medium beginning suddenly at a height h and extending to infinity. For waves with polarization both perpendicular to the plane of incidence (from a horizontal antenna) and in the plane of incidence (from a vertical antenna), show from Fresnel's equations or reflection and refraction that for $\omega > \omega_p$ there is a range of angles of incidence for which reflection is not total, but for larger angles there is a total reflection back toward the earth.

c) A radio amateur operating at a wavelength of 21 meters in the early evening finds that she can receive distant stations located more than 1000 km away, but none closer. Assuming that the signals are being reflected from the F layer of the ionosphere at an effective height of 300 km, calculate the electron density. Compare with the known maximum and minimum F layer densities of *ca.* $2 \times 10^6 \text{ cm}^{-3}$ in the daytime and *ca.* $(2 - 4) \times 10^5 \text{ cm}^{-3}$ at night.

4. Skin effect.

Consider a linearly polarized wave with frequency ω normally incident on a metal described by Ohmic conductivity σ . Show that this problem can be analyzed by treating the metal as a 'generalized dielectric material' with complex $\epsilon(\omega) = 1 + i(4\pi\sigma/\omega)$.

a) Find the wavenumber of the EM wave within the metal. Show that the wave amplitude decreases exponentially as a function of distance from the surface. Relate the EM field penetration depth (called skin length) to the imaginary part of the wavenumber. Estimate the skin length for a good metal in the optical frequency range, using a typical value of $\sigma \sim 10^{17} \text{ s}^{-1}$.

b) Estimate plasma frequency of a metal, using a typical electron density $n \sim 10^{23} \text{ cm}^{-3}$. Find the reflection coefficient of a metal, described by $\epsilon(\omega) = 1 - \omega_p^2/\omega(\omega + i\gamma)$. Sketch the frequency dependence $R(\omega)$. Estimate the value of R for the parameter values used in part a)

5. Group velocity.

Consider waves in a homogeneous dielectric characterized by an index of refraction $n(\omega) = \sqrt{\epsilon(\omega)}$.

a) Show that the general solution for plane waves in one dimension can be written as

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} [A(\omega)e^{i\kappa(\omega)x} + B(\omega)e^{-i\kappa(\omega)x}], \quad \kappa(\omega) = (\omega/c)n(\omega) \quad (2)$$

where $u(x, t)$ is a component of \mathbf{E} or \mathbf{B} .

b) To study time evolution of a wavepacket, consider a gaussian $A(\omega) = A_0 e^{-\frac{1}{2}\tau_0^2(\omega-\omega_0)^2}$, which describes a superposition of waves with frequencies in a narrow band around $\omega = \omega_0$. Evaluate the integral over ω at $x = 0$ (with $B(\omega) = 0$) and show that in the time domain it corresponds to a packet with base frequency ω_0 and a gaussian envelope of width *ca.* τ_0 .

c) To analyze wavepacket propagation, expand $\kappa(\omega)$ up to linear terms about $\omega = \omega_0$,

$$\kappa(\omega) = \kappa(\omega_0) + \frac{1}{v_g}(\omega - \omega_0) + O((\omega - \omega_0)^2), \quad v_g = d\omega/d\kappa \quad (3)$$

Evaluate the integral over ω and show that the wavepacket propagates with velocity v_g , known as *group velocity*. How does the wavepacket envelope evolve in time?

6. Surface EM waves.

Consider an interface between two dielectrics, one of which has negative dielectric constant. Study the waves that can propagate along the interface. Show that there is a TM (transverse magnetic) wave in which the magnetic field is parallel to the interface and normal to the wave propagation direction. The spatial z dependence of the magnetic field can be described as an *evanescent* wave, decreasing exponentially as a function of distance from the interface,

$$B(x, z)_{z>0} = B_0 e^{ikx - \kappa_1 z}, \quad \kappa_1 = \sqrt{k^2 - \frac{\omega^2}{c^2}\epsilon_1}, \quad (4)$$

$$B(x, z)_{z<0} = B_0 e^{ikx + \kappa_2 z}, \quad \kappa_2 = \sqrt{k^2 + \frac{\omega^2}{c^2}|\epsilon_2|}, \quad (5)$$

where $\epsilon_1 > 0$, $\epsilon_2 < 0$. Here z is the coordinate normal to the interface, x and y are the coordinates in the interface plane, x is the propagation direction, and the field is in the y direction.

a) Show that the wave (4) can only exist if $\epsilon_1 < |\epsilon_2|$. Find the dispersion relation $\omega(k)$ of the wave.

b) The condition $\epsilon_1 < |\epsilon_2|$ can be realized at the metal-vacuum interface for frequencies slightly below plasma frequency of the metal. Assuming that the metal is described by the plasma model (1), find the dispersion relation of the surface plasma EM wave.