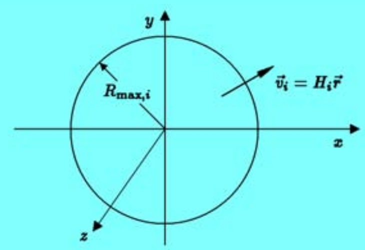
8.286 Lecture 8
October 1, 2013

DYNAMICS OF HOMOGENEOUS EXPANSION, PART IV

Summary: Mathematical Model



$$\begin{aligned} t_i &\equiv \text{time of initial picture} \\ R_{\max,i} &\equiv \text{initial maximum radius} \\ \rho_i &\equiv \text{initial mass density} \\ \vec{v}_i &= H_i\vec{r}\ . \end{aligned}$$

Summary: Equations

Want: $r(r_i, t) \equiv$ radius at t of shell initially at r_i

Find: $r(r_i, t) = a(t)r_i$, where

$$\text{Friedmann Equations}\begin{cases} \ddot{a} = -\dfrac{4\pi}{3}G\rho(t)a \\ H^2 = \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi}{3}G\rho - \dfrac{kc^2}{a^2} \quad \text{(Friedmann Eq.)} \end{cases}$$

and

$$\rho(t) \propto \frac{1}{a^3(t)}\ , \text{ or } \rho(t) = \left[\frac{a(t_1)}{a(t)}\right]^3 \rho(t_1) \text{ for any } t_1.$$

Units: $[r] =$ meter, $[r_i] =$ notch, $[a(t)] =$ m/notch, $[k] = 1/\text{notch}^2$.

Summary: Conventions

Us: Notch is arbitrary (free to be redefined each time we use it).

[Our construction used $a(t_i) = 1$ m/notch, but we can interpret this equation as the definition of t_i. But we can forget the definition of t_i if we don't intend to use it.]

Ryd $a(t_0) = 1$ (where $t_0 =$ now). For us, $1 \to 1$ m/notch.

Many Other Books: if $k \neq 0$, then $k = \pm 1$.

(For us, $\pm 1 \to \pm 1$ m/notch.)

Summary: Types of Solutions

$$\boxed{\dot{a}^2 = \frac{8\pi G}{3}\frac{\rho(t_1)a^3(t_1)}{a(t)} - kc^2\ .}$$

For intuition, remember that $k \propto -E$, where E is a measure of the energy of the system.

Types of Solutions:

1) $k < 0$ ($E > 0$): unbound system. $\dot{a}^2 > (-kc^2) > 0$, so the universe expands forever. **Open Universe.**

2) $k > 0$ ($E < 0$): bound system. $\dot{a}^2 \geq 0 \quad \Longrightarrow$

$$\boxed{a_{\max} = \frac{8\pi G}{3}\frac{\rho(t_1)a^3(t_1)}{kc^2}\ .}$$

Universe reaches maximum size and then contracts to a Big Crunch. **Closed Universe.**

3) $k = 0$ ($E = 0$): critical mass density.

$$H^2 = \frac{8\pi G}{3}\rho - \underbrace{\frac{kc^2}{a^2}}_{=0} \quad \Longrightarrow \quad \boxed{\rho \equiv \rho_c = \frac{3H^2}{8\pi G}\ .}$$

Flat Universe.

Summary: $\rho > \rho_c \Longleftrightarrow$ closed, $\rho < \rho_c \Longleftrightarrow$ open, $\rho = \rho_c \Longleftrightarrow$ flat.

Numerical value: For $H = 67.3$ km-s^{-1}-Mpc^{-1} (Planck 2013 plus other experiments),

$$\rho_c = 8.4 \times 10^{-30}\ \mathrm{g/cm}^3 \approx 5 \text{ proton masses per m}^3.$$

Definition: $\Omega \equiv \dfrac{\rho}{\rho_c}$.

Summary: Evolution of a Flat Universe

If $k = 0$, then

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{\text{const}}{a^3} \quad \Longrightarrow \quad \frac{da}{dt} = \frac{\text{const}}{a^{1/2}}$$

$$\Longrightarrow \quad a^{1/2} a\, da = \text{const}\, dt \quad \Longrightarrow \quad \frac{2}{3}a^{3/2} = (\text{const})t + c'\ .$$

Choose the zero of time to make $c' = 0$, and then

$$\boxed{a(t) \propto t^{2/3}\ .}$$