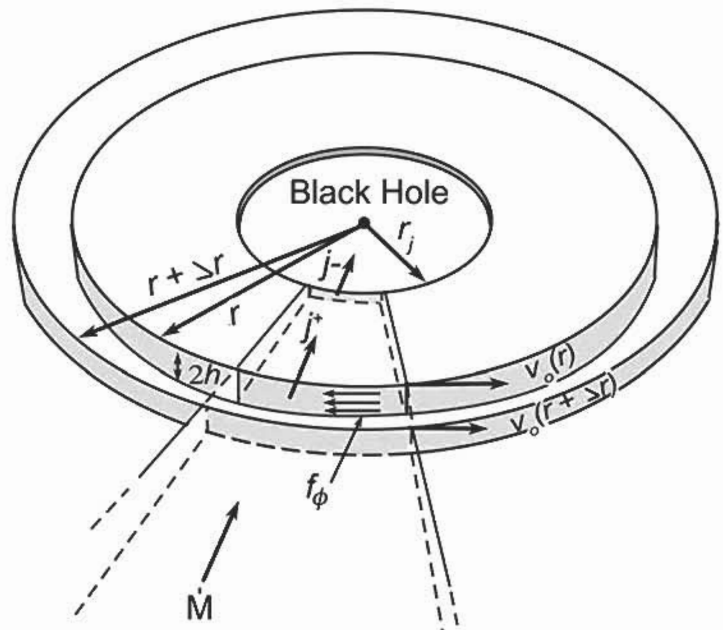
Steady State Accretion Disk Model
(Shakura & Sunyaev 1973)

We seek a steady-state solution to matter accreting at a constant rate $\dot{M}$ onto a central collapsed star via a geometrically thin accretion disk. The figure below, from Shapiro & Teukolsky's book, indicates the geometry of such a disk.



Slice of a thin, Keplerian accretion disk around a central black hole.

As in most fluid flow problems there are (at least) 6 equations describing the physics. These are

Conservation of Mass
Conservation of Momentum (3 components)
Conservation of Energy
Equation of Radiative Transport

There is also, of course, an equation of state that needs to be specified.

First, a general word about the fluid equations – a generic conservation law for a physical quantity x looks like:

$$\frac{\partial n_x}{\partial t} + \vec{\nabla}\cdot(n_x \vec{v}) = \text{Source term} \qquad (1)$$

where n_x is the density of quantity x, and $\vec{v}$ is the flow velocity. In spherical coordinates, for a pure radial flow,

$$\vec{\nabla}\cdot \quad = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \quad)$$

while in cylindrical coordinates, for a radial flow,

$$\vec{\nabla}\cdot \quad = \frac{1}{r}\frac{\partial}{\partial r}(r \quad).$$

In the case of an accretion disk, the geometry is only approximately cylindrical, i.e., the radial motion of the fluid is not completely in the x y plane ($\hat{z}$ is the rotation axis). This can be understood from the fact that the disk thickness is not generally constant with radial distance r. In such cases, it is often handy to use a generalization of the conservation equations of the form:

$$\frac{\partial n_x}{\partial t} + \frac{1}{A}\frac{\partial}{\partial r}(A n_x v_r) = \text{Source term} \qquad (2)$$

where A expresses the way in which the area (perpendicular to r) increases with radial distance. Finally, for motion which is <u>nearly</u> Keplerian at every point in the disk, the 3 equations of conservation of momentum are better expressed as conservation of angular momentum and vertical ($\hat{z}$) linear momentum, along with the explicit assumption that

$$\Omega^2 \cong \frac{GM}{r^3} \qquad (3)$$

everywhere in the disk, where M is the mass of the central object, and Ω is the angular velocity.

For the steady-state solution we are seeking, all the time derivatives drop out and we are left with:

<u>Conservation of Mass</u>: $$\frac{1}{2\pi r H}\frac{d}{dr}\left(2\pi r H \rho v_r\right) = 0 \qquad (4)$$

where $H(r)$ is the disk thickness at r, and ρ is the mass density, which depends on both r and z. Actually, the professionals who work on disks define $\int_{-\infty}^{\infty}\rho\, dz \equiv \rho_0 H \equiv \Sigma_1$, where Σ_1 is the column thickness through the disk, and ρ_0 is the midplane density; to keep things in more familiar terms I will continue to use ρ and H separately.

<u>Angular Momentum</u>: $$\frac{1}{2\pi r H}\frac{d}{dr}\Big[2\pi r H \underbrace{(\rho \Omega r^2)}_{\text{angular momentum density}} v_r\Big] = \underset{\substack{\uparrow \\ \text{torque per unit volume}}}{T} \qquad (5)$$

<u>Vertical Pressure Balance</u>: $$\frac{dP}{dz} = -g_z \rho = -\frac{GMz\rho}{r^3} \qquad (6)$$

For conservation of energy, we simply assume that all the energy generated locally in the disk via viscous dissipation is transported vertically ($\hat{z}$ direction) and radiated away at the surface as black body radiation. For this to be true, the disk must be optically thick even though it is geometrically thin.

<u>Conservation of Energy</u>: $$\sigma T_{eff}^4 = \dot{Q} \qquad (7)$$

where $\dot{Q}$ is the energy dissipation in the disk per unit area perpendicular to the disk.

Radiative Transport:
$$\frac{4}{3}\frac{\sigma T^4}{\tau} = \sigma T_{eff}^4 \qquad (8)$$

where T is the temperature in the disk midplane and τ is the optical thickness through the disk $\tau \equiv \int_{-\infty}^{\infty} \rho \kappa \, dz$. If τ is large $T >> T_{eff}$. Equation (8) can be derived from the first moment of the equation of radiative transport that we derived for use in stellar interiors. Recall

$$c\,\frac{dP_{rad}}{dz} + \rho \kappa F = 0$$

$$F = \frac{c}{\rho\kappa}\left(\frac{dP_{rad}}{dz}\right) = \frac{4}{3}\sigma\left(\frac{dT^4}{dz}\right)\frac{1}{\rho\kappa}$$

The vertical structure of the disk is usually treated in the "single zone" approximation. Thus,

$$F \simeq \frac{4}{3}\sigma\frac{1}{\rho\kappa}\frac{(T^4 - T_{eff}^4)}{H} \simeq \frac{4\sigma}{3\tau}T^4 \qquad (8a)$$

Equation of State: In general, the equation of state can be pretty complicated; however, in order to obtain analytic solutions for the Shakura-Sunyaev disk model, we consider two limits of perfect gases:

(i) $P = \rho kT/\mu$ (gas pressure dominated) (9a)

(ii) $P = \frac{1}{3}aT^4$ (radiation pressure dominated) (9b)

<u>Opacities</u>: Opacities are also typically messy functions of ρ and T; however, to obtain analytic solutions for the S-S disk, we usually stick to either

(i) Electron scattering; κ_T, independent of ρ and T (10a)

(ii) Free-free absorption; Kramer's opacity $\propto \rho T^{-3.5}$ (10b)

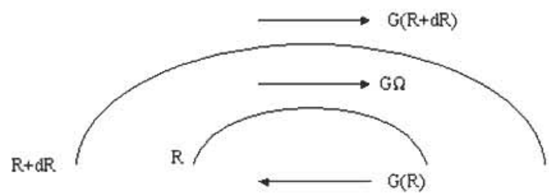
Two crucial terms that have yet to be evaluated are $\mathcal{T}$ in equation (5) and $\dot{Q}$ in equation (7). To develop a specific expression for these quantities we need to estimate the torques and the energy dissipated due to viscosity in the disk. In the absence of viscosity (or other external torques) the disk would stagnate, and no matter would flow onto the central object. To compute $\mathcal{T}$ and $\dot{Q}$ we need an explicit expression for the viscous shear stress $t_{r\phi}$. This is the force on the fluid in the ϕ direction per unit area in the radial direction.

At the time of the S-S paper (1973) the thing that was most clear about the form of the viscosity law was that it was <u>not</u> due to molecular viscosity (which was too small by orders of magnitude). The dominant contributions to the viscosity probably result from turbulence and/or magnetic fields. To this day, it is still controversial as to how to formulate viscosities in accretion disks. S-S suggested a simple, plausible, dimensional argument which was that

$$t_{r\phi} = \alpha P \tag{11}$$

where P is the local pressure (which has the same dimensions), and α is a dimensionless, constant free parameter. This prescription is still widely used today.

To compute $\mathcal{T}$ (in equation 5), consider the forces acting on the "walls" of an annular ring within the disk.



Differential viscous torque.

The matter just interior to the annulus tends to speed the ring up, while the matter just outside the annulus tends to slow the ring down. The net torque on the ring per unit volume is:

$$\mathcal{T} = \frac{+\alpha P_o (2\pi r_o H_o) r_o - \alpha \overbrace{P_i}^{\text{stress}} \overbrace{(2\pi r_i H_i)}^{\text{area}} \overbrace{r_i}^{\text{lever arm}}}{2\pi r H \Delta r} \qquad (12)$$

where the subscripts $\underline{i}$ and $\underline{o}$ denote values at r and $r+\Delta r$, respectively. Thus,

$$\mathcal{T} = \frac{\alpha}{Hr}\frac{d}{dr}\left(P r^2 H\right) \qquad (12a)$$

Similarly, the power dissipation per unit area is given by the torque multiplied by the <u>relative</u> angular velocity between the edge and the center of the annulus:

$$\dot{Q} = \frac{-\overbrace{\left(\alpha P 2\pi r^2 H\right)}^{\text{torque}} \overbrace{\left(\frac{d\Omega}{dr}\right)\Delta r}^{\Delta\Omega}}{\underbrace{2}_{\text{top and bottom of disk}} \cdot 2\pi r \Delta r} \qquad (13)$$

Therefore $\dot{Q} = \frac{3}{4} \alpha H P \Omega$ (14)

Let us now solve the equation for conservation of mass. Equation (4) becomes simply

$$2\pi r H \rho v_r = \text{constant} \equiv \dot{M} \qquad (15)$$

where $\dot{M}$ is the rate of mass passing through the disk and onto the central object.

We can also solve equation (5) for angular momentum conservation.

$$\frac{1}{2\pi r H} \frac{d}{dr}\left[2\pi r H \rho \Omega r^2 v_r\right] = \frac{1}{2\pi r H}\frac{d}{dr}\left[\dot{M}\Omega r^2\right] = \frac{\alpha}{Hr}\frac{d}{dr}\left[P r^2 H\right]$$

where the middle expression was obtained by substituting the identity from equation (15). The solution to equation (16) is trivial and yields (16)

$$\dot{M}\Omega r^2 = 2\pi\alpha P r^2 H + \text{constant} \qquad (17)$$

Let us assume that at the inner boundary of the disk (at r_I; see Figure 14.3) the pressure vanishes (e.g., this might occur at the innermost stable orbit around a black hole). This provides the constant of integration $= \dot{M}\Omega(r_I) r_I^2$.

$$PH = \frac{\dot{M}\left(\Omega r^2 - \Omega_I r_I^2\right)}{2\pi\alpha r^2} = \frac{\dot{M}\Omega\left(1 - \frac{r_I^{1/2}}{r^{1/2}}\right)}{2\pi\alpha} \qquad (18)$$

So, $$\boxed{PH = \frac{\dot{M}\Omega f^4}{2\pi\alpha}} \qquad (18a)$$

where $f(r) \equiv \left(1 - r_I^{1/2}/r^{1/2}\right)^{1/4}$. Thus, the first major result to emerge from the analysis is that, in order to transport mass through the disk at a constant rate $\dot{M}$, the product PH must vary with radial distance as indicated by equation (18a).

Next, we combine equations (7), (14) and (18a) to find

$$\sigma T_{eff}^4 = \dot{Q} = \frac{3}{4} \alpha H P \Omega = \frac{3}{8} \frac{\dot{M} \Omega^2 f^4}{\pi} \qquad (19)$$

$$\text{or} \quad T_{eff}^4 = \frac{3 G M \dot{M}}{8 \pi \sigma r^3} \left(1 - r_I^{1/2}/r^{1/2}\right) \qquad (19a)$$

$$\text{or} \quad \boxed{T_{eff} = \left(\frac{3G}{8\pi\sigma}\right)^{1/4} M^{\frac{1}{4}} \dot{M}^{1/4} r^{-3/4} f} \qquad (19b)$$

Remarkably, this result for the effective temperature of the disk is <u>independent</u> of the unknown parameter α.

There is only one differential equation which remains to be solved before we have reduced the disk equations to four algebraic equations which can be solved for P, ρ, T, and H. Equation (6) which requires hydrostatic equilibrium in the vertical direction leads to

$$P = +\frac{GM}{r^3} \int_0^\infty \rho(z) z^2 dz \cong + \frac{GM}{r^3} \rho H^2 , \qquad (20)$$

in keeping with the single-zone approximation to the vertical structure of the disk.

To go any further, we must select an equation of state and an opacity law. To illustrate how this works, choose

$$P = \rho k T/\mu \qquad (9a)$$

$$K = K_0 \rho T^{-3.5} \qquad (10b)$$

Summarizing all the relevant algebraic equations, we have

1. $T_{eff} = \left(\frac{3G}{8\pi}\right)^{1/4} M^{1/4} \dot{M}^{1/4} r^{-3/4} f$ (19b)

2. $PH = \frac{\sqrt{GM}}{2\pi} \dot{M} r^{-3/2} f^4 \alpha^{-1}$ (18a)

3. $T^4 = \frac{3\kappa \rho H T_{eff}^4}{4}$ (8)

$\kappa = \kappa_e$ or $\kappa_0 \rho T^{-3.5}$ (10a or 10b)

4. $P = \rho k T/\mu$ or $P = \frac{1}{3} a T^4$ (9a or 9b)

5. $P = GM\rho H^2 r^{-3}$ (20)

<u>Gas Pressure Dominated / Kramer's opacity</u>

The solution to 1 through 5 yields

$$T_{eff} \propto \dot{M}^{1/4} r^{-3/4} f$$

$$T \propto \dot{M}^{3/10} \alpha^{-1/5} r^{-3/4} f^{6/5}$$

$$\rho \propto \dot{M}^{11/20} \alpha^{-7/10} r^{-15/8} f^{11/5}$$

$$P \propto \dot{M}^{17/20} \alpha^{-9/10} r^{-21/8} f^{17/5}$$

$$H \propto \dot{M}^{3/20} \alpha^{-1/10} r^{9/8} f^{3/5}$$

The proportionality constants are included in the enclosure from the Frank, King, & Raine book.

Radiation Pressure / Kramer's opacity

$$T \propto \dot{M}^{4/25} \alpha^{-6/25} r^{-3/5} f^{16/25}$$

$$\rho \propto \dot{M}^{-2/25} \alpha^{-22/25} r^{-6/5} f^{-8/25}$$

$$P \propto \dot{M}^{16/25} \alpha^{-24/25} r^{-12/5} f^{64/25}$$

$$H \propto \dot{M}^{9/25} \alpha^{-1/25} r^{9/10} f^{36/25}$$

Radiation Pressure / Electron Scattering

$$T \propto \dot{M}^{0} \alpha^{-1/4} r^{-3/8} f^{0}$$

$$\rho \propto \dot{M}^{-2} \alpha^{-1} r^{3/2} f^{-8}$$

$$P \propto \dot{M}^{0} \alpha^{-1} r^{-3/2} f^{0}$$

$$H \propto \dot{M} \alpha^{0} r^{0} f^{4}$$

Gas Pressure / Electron Scattering

$$T \propto \dot{M}^{2/5} \alpha^{-1/5} r^{-9/10} f^{8/5}$$

$$\rho \propto \dot{M}^{2/5} \alpha^{-7/10} r^{-33/20} f^{8/5}$$

$$P \propto \dot{M}^{4/5} \alpha^{-9/10} r^{-51/20} f^{16/5}$$

$$H \propto \dot{M}^{1/5} \alpha^{-1/10} r^{21/20} f^{4/5}$$

Some final notes:

1. The S-S disk models represent steady-state, thermal equilibrium models. However, it is important to note that these solutions are not stable for all choices of equation of state. In particular, they become

unstable for transitions between gas and radiation pressure dominated regimes, and in regions where there is a transition between ionized and neutral gas.

2. The type of disk instability mentioned in (1) probably results in the type of quasi regular outbursts observed in cataclysmic variables and black hole transient X-ray sources.

3. In the derivations above, we expressed the viscous stress as $t_{r\varphi} = \alpha P$, without explicitly writing it in terms of a viscosity coefficient. The connection is

$$t_{r\varphi} = \frac{3}{2}\eta\Omega \equiv \alpha P$$

where η is the coefficient of dynamic viscosity ($g\,cm^{-1}sec^{-1}$).

4. The last 3 pages of this handout are taken from the Frank, King, & Raines book. They give a derivation of the viscous torque and viscous dissipation terms that complements the one given earlier in this writeup.