Lecture 9 - Topics

- Change of variable

1. Change of Variables, 1 Variable

$$\int f(x)dx = \int \widetilde{f}(u)\frac{dx(u)}{du}du$$

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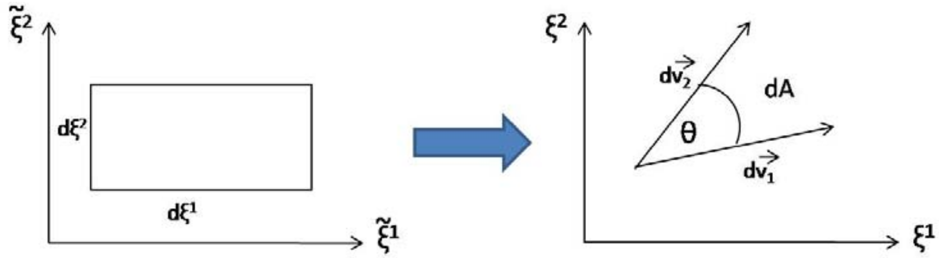
$u = u(x)$. Assume invertible function $x(u)$

$$\widetilde{f}(u) = f(x(u))$$

2 Variables of Integration

$$\int f(\xi^1, \xi^2)d\xi^1\xi^2$$

Let $M_{ij} = \frac{\partial \xi^i}{\partial \widetilde{\xi}^j}$



$$dv_1^i = \frac{\partial \xi^i}{\partial \widetilde{\xi}^1}d\widetilde{\xi}^1$$

$$dv_2^i = \frac{\partial \xi^i}{\partial \widetilde{\xi}^2}d\widetilde{\xi}^2$$

$$f(\xi^1, \xi^2)d\xi^1\xi^2 = f(\xi^1, \xi^2)dA$$

$$\begin{aligned} dA &= |d\vec{v}_1||d\vec{v}_2|\sin\theta \\ &= \sqrt{|dv_1||dv_2| - (d\vec{v}_1 \cdot d\vec{v}_2)} \\ &= \sqrt{\left(\frac{\partial \xi^i}{\partial \widetilde{\xi}^1}\frac{\partial \xi^i}{\partial \widetilde{\xi}^1}\right)\left(\frac{\partial \xi^j}{\partial \widetilde{\xi}^2}\frac{\partial \xi^j}{\partial \widetilde{\xi}^2}\right) - \left(\frac{\partial \xi^i}{\partial \widetilde{\xi}^1}\frac{\partial \xi^i}{\partial \widetilde{\xi}^2}\right)^2}\, d\widetilde{\xi}^1 d\widetilde{\xi}^2 \end{aligned}$$

$$dA = \sqrt{(M_{i1} \cdot M_{i1})(M_{j2} \cdot M_{j2}) - M_{i1}M_{i2}M_{j1}M_{j2}} d\widetilde{\xi^1}\widetilde{\xi^2}$$

$$M_{1i} = (M^T)_{i1}$$

$$dA = \sqrt{(M^TM)_{11}(M^TM)_{22} - (M^TM)_{12}^2} d\widetilde{\xi^1}d\widetilde{\xi^2} = \sqrt{det(M^TM)} d\widetilde{\xi^1}d\widetilde{\xi^2}$$

$$det(M^TM) = det(M^T)det(M)$$

$$dA = |det(M)| d\widetilde{\xi^1}d\widetilde{\xi^2}$$

So:

$$f(\xi^1, \xi^2)d\xi^1 d\xi^2 = \widetilde{f}(\widetilde{\xi^1}, \widetilde{\xi^2})|det(\frac{\partial \xi^i}{\partial \xi^j})| d\widetilde{\xi^1}d\widetilde{\xi^2}$$

The goal is to verify: $A = \int d\xi^1 d\xi^2 \sqrt{g}$ where $g = det(g_{ij})$ is reparam. invariant.

$$\begin{aligned} g_{ij}(\xi) \cdot d\xi^i d\xi^j &= \widetilde{g}_{pq}(\widetilde{\xi}) d\widetilde{\xi}^p d\widetilde{\xi}^q \\ &= \widetilde{g}_{pq}(\widetilde{\xi}) \left(\frac{\partial \widetilde{\xi}^p}{\partial \xi^i}\right)\left(\frac{\partial \widetilde{\xi}^q}{\partial \xi^j}\right) d\xi^i d\xi^j \end{aligned}$$

Let $\widetilde{M}_{ij} = \frac{\partial \widetilde{\xi}_i}{\partial \xi_j}$

$$\begin{aligned} g_{ij}(\xi) &= \widetilde{g}_{pq}\widetilde{M}_{pi}\widetilde{M}_{pj} \\ &= (\widetilde{M}^T)_{ip}\widetilde{g}_{pq}\widetilde{M}_{qj} \\ &= (\widetilde{M}^T\widetilde{g}\widetilde{M})_{ij} \end{aligned}$$

$$g = det(g_{ij}) = det(\widetilde{M}^T)\widetilde{g}det(\widetilde{M}) = \widetilde{g}|det(\widetilde{M})|^2$$

$$det(\widetilde{M}^T) = det(\widetilde{M})$$

$$A = \int d\xi^1 d\xi^2 \sqrt{g} = \int d\widetilde{\xi}^1 d\widetilde{\xi}^2 det(u)\sqrt{\widetilde{g}} det(\widetilde{M})$$

$$(M\widetilde{M})_{ij} = M_{ik}\widetilde{M}_{ki} = \frac{\partial \xi^i}{\partial \widetilde{\xi}^k}\frac{\partial \widetilde{\xi}^k}{\partial \xi^j} = \frac{\partial \xi^i}{\partial \xi^j}$$

If $i \neq j$, this equals 0. If $i = j$, this equals 1. Therefore, we have δ^i_j.

$$det(M) = det(\widetilde{M})$$

$$A = \int d\widetilde{\xi^1} d\widetilde{\xi^2} \sqrt{\widetilde{g}}$$

Goal: Write area functional for spacetime surface. Just did this for a surface in Euclidean space. Now do for a surface in Mintowsk space (so there's a negative sign instead of all positive signs).

Change of notation: $(\xi^1, \xi^2) \to (\tau, \sigma)$ where τ is "like time" and σ is "like time".

Target space:

$$x^\mu = (x^0, x^1, \ldots, x^d)$$

$D = d + 1 =$ space time dimension. $d =$ spatial dimension.

Mapping:

$$x^\mu(\tau, \sigma) = X^\mu(\tau, \sigma)$$

X^μ sometimes called "string coordinates". σ has a finite range. For a closed string, periodic. τ can have an infinite range.

Area constructed from: $dv_1^\mu = \frac{dX^\mu}{d\tau} d\tau$, $dv_2^\mu = \frac{dX^\mu}{d\sigma} d\sigma$.

By analogy: $dA = \sqrt{(dv_1 \cdot dv_1)(dv_2 \cdot dv_2) - (dv_1 \cdot dv_2)^2}$?

The problem is that the number under the square root is less than 0, and we don't want an imaginary dA!

Static String:

$$X^0(\tau, \sigma) = c\tau$$
$$X^i(\tau, \sigma) = f^i(\sigma)$$

dv_1^μ has only $\mu = 0$ component.
dv_2^μ has only $\mu \neq 0$ components.

$$dv_1 \cdot dv_1 < 0$$
$$dv_2 \cdot dv_2 > 0$$
$$dv_1 \cdot dv_2 = 0$$

Therefore:

$$dA = \sqrt{< 0}$$

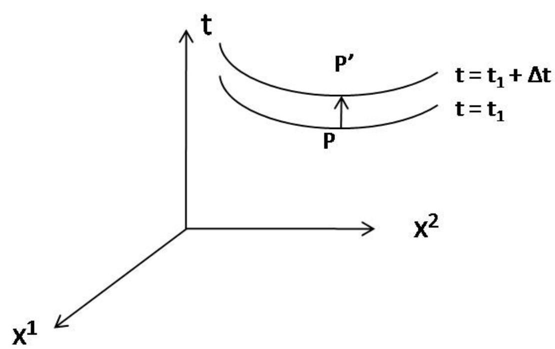
So instead:

$$dA = d\tau d\sigma \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \tau}\right)\left(\frac{\partial X}{\partial \sigma} \cdot \frac{\partial X}{\partial \sigma}\right)}$$
$$= d\tau d\sigma \sqrt{\left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \sigma}\right)^2 - \left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \tau}\right)\left(\cdots\right)}$$

Consider worldline $x^\mu(\tau)$: $\frac{dx^\mu}{d\tau}$ is timelike. Particle moves slower than light.

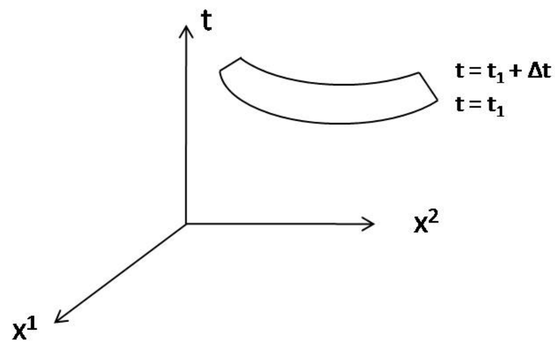
Consider point P on worldsheet of a string. Worldsheet described by τ and σ so:

$$X_p = X^\mu(\tau_P, \sigma_P)$$



If all points P on string $\exists$ a point P' on string at time Δt. $u_{P',P}$ is timelike then string moving slower than light.

The worldsheet is the area swept out by the string over time.



1. $\forall P \exists$ spacelike tangent
2. Just saw $\forall P \exists$ timelike tangent too.

Will use 1 and 2 to show $dA = \sqrt{> 0}$.

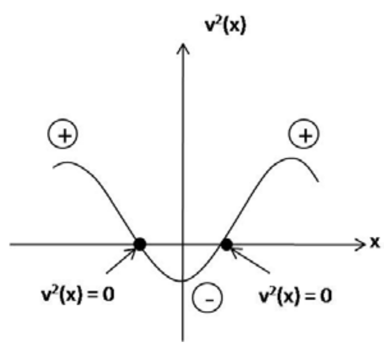
Consider tangent vectors at P spanned by $\frac{\partial X^\mu}{\partial \tau}(P), \frac{\partial X^\mu}{\partial \sigma}(P)$.

Consider 1 parameter family of vectors.

$$v^\mu(\lambda) = \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma}$$

Linear combination of $\partial X^\mu / \partial \tau$ and $\partial X^\mu / \partial \sigma$ with coefficients 1&λ. Most agreed would have 2 arbitrary coefficients, but here only care about direction.

$$v^2(\lambda) = \lambda^2 \left(\frac{\partial X}{\partial \sigma} \right)^2 + \partial x \left(\frac{\partial X}{\partial \sigma} \cdot \frac{\partial X}{\partial \tau} \right) + \left(\frac{\partial X}{\partial \tau} \right)^2$$



Get quadratic equation for λ

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac \leq 0$ will get complex (not real) roots. So $b^2 - 4ac > 0$.