Coordinates and Proper Time

1. Introduction

These notes supplement Chapter 1 of EBH (*Exploring Black Holes* by Taylor and Wheeler). They elaborate on the discussion of bookkeeper coordinates and how coordinates are related to actual physical distances and times. Also, a brief discussion of the classic Twin Paradox of special relativity is presented in order to illustrate the principal of maximal (or extremal) aging.

Before going to details, let us review some jargon whose precise meaning will be important in what follows. You should be familiar with these words and their meaning. Spacetime is the four-dimensional playing field for motion. An event is a point in spacetime that is uniquely specified by giving its four coordinates (e.g. t, x, y, z). Sometimes we will ignore two of the spatial dimensions, reducing spacetime to two dimensions that can be graphed on a sheet of paper, resulting in a Minkowski diagram. A worldline is a continuous, one-dimensional curve in spacetime. Generally speaking, we only use the word "worldline" when referring to the path taken by a massive particle, which is restricted by relativity to travelling slower than the speed of light in vacuum. We will always choose units so that $c = 1$; see EBH Section 1-2.

A reference frame is defined by a set of spatial axes with a specified motion. For example, the reference frame of the Earth consists of a set of axes centered on the center of the Earth, so that the Earth is always at rest in this frame. Reference frames may be global or local depending on whether they are used to describe motion over all of spacetime or only close to the center of the frame, respectively. In Newtonian mechanics and special relativity, but not in general relativity, there exists the concept of a global inertial frame in which Newton's laws of mechanics apply everywhere. As we will see, this is no longer the case in general relativity. Note that we often drop the word "reference" from "reference frame".

Reference frames are closely related to coordinate systems. Coordinate systems assign a set of labels to each event in spacetime (or point in space, if we are doing ordinary geometry without treating time). Coordinate systems are *very important* in general relativity. They provide an unambiguous way to deal with motion and geometry even when global inertial reference frames no longer exist.

For these reasons, in general relativity we will use coordinate systems instead of reference frames to provide the global description of motion in spacetime. However, local reference frames are very useful for interpreting motion as seen in the vicinity of an observer. Since we are all observers, we must learn how to use both global coordinate systems and local reference frames.

2. Bookkeeper Coordinates in Special Relativity

We are used to thinking of coordinates like (t, x, y, z) as having direct, physical significance, i.e. as being measurable quantities. However, special relativity teaches us that the results of measurement depend on the observer's motion. There are no universal coordinates that give the distances and times measured by all observers. Absolute space and time are a myth.

Special relativity has a special class of reference frames known as global inertial reference frames. Coordinates in an inertial reference frame give physical times and distances as measured by an observer at rest in that frame. The coordinates of different inertial observers are related by Lorentz transformations. For example, a Lorentz boost in the x-direction relates coordinates in the two different frames as follows

$$t' = \gamma(t - vx) \ , \ x' = \gamma(x - vt) \ , \ y' = y \ , \ z' = z \ , \tag{1}$$

where v is the relative velocity of the two frames and $\gamma \equiv (1 - v^2)^{-1/2}$. There are many other possible Lorentz transformations (e.g., boosts along the y- or z-directions, and spatial rotations), but this example is enough to make the point: coordinates are arbitrary labels assigned by us for our convenience in locating events. EBH reinforces this point by calling them bookkeeper coordinates.

Lorentz transformations have the very important property of leaving invariant a quantity called the interval. If two spacetime events have coordinates (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) respectively, then the spacetime interval (in special relativity only) is

$$(\Delta\tau)^2 = (t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2 \ . \tag{2}$$

If you've had special relativity, you recognize this as the square of the proper time difference $\Delta\tau$ between the two spacetime events. (The proper time — often called wristwatch time in EBH — is the physical time in an inertial reference frame in which both events are simultaneous.) Sometimes the interval is defined to be the negative of this, or the square root; what is important is not the sign or the root but the fact that this combination is unchanged if the coordinates are Lorentz transformed. (You can easily check this by replacing t_1 with t'_1, t_2 with t'_2, etc., and substituting eq. 1.) In fact, the Lorentz transformation is *defined* to be any linear transformation that leaves the interval invariant.

When the two spacetime points (events) are taken very close to each other, the spacetime interval becomes the metric. *The metric is a formula telling how infinitesimal increments of proper time or proper distance are related to coordinate differentials.* If the coordinate differences are written (dt, dx, dy, dz), the metric is

$$d\tau^2 = dt^2 - (dx^2 + dy^2 + dz^2) \ . \tag{3}$$

Please note the unusual notation: $d\tau^2$ is an abbreviation for $(d\tau)^2$ and not $d(\tau^2)$, and similarly for the other terms in equation (3). This notation is universally used in general relativity.

The spatial part of the interval is recognizable from the Pythagorean theorem as the square of the distance. As a physical quantity (the distance), it is, of course, independent of spatial rotations of the coordinates. Einstein taught us that this invariance of space extends to an invariance of *spacetime* with time as the fourth dimension (conventionally we list time as the first dimension). The implications of this brilliant extension are the subject matter of special relativity, which we hope you have studied either in a class or on your own.

In one way, general relativity is simply the extension of special relativity to work with arbitrary coordinate transformations. Special relativity uses Cartesian coordinates and the time measured in some inertial reference frame. General relativity is, as its name suggests, more general.

3. Coordinate Transformations

General relativity drops the assumption that the spatial coordinates are Cartesian and the time coordinate measures physical time for the simple reason that such coordinates are impossible in curved spacetime. (More on this in Chapter 2 of EBH; for now, please take our word for it.) We must therefore consider coordinates on curved spaces. A smooth space of any number of dimensions is called a manifold. According to general relativity, spacetime is a four-dimensional curved manifold. There is a very important theorem of geometry: it is impossible to set up a global Cartesian coordinate system, for which the Pythagorean theorem (or its extension to include time) holds, in a curved manifold.

Even in flat spaces, where Cartesian coordinates can be used, non-Cartesian coordinates are allowed. For example, in many applications it is more convenient to use polar instead of Cartesian coordinates. It is therefore useful to review how various coordinates transform.

We can illustrate coordinate transformations graphically. Figure 1 shows a section of the plane with four different coordinate systems. To make the discussion more interesting, we take the plane to be space-time rather than space-space; however the same concept of coordinate transformations would apply equally well if this were the x-y plane.

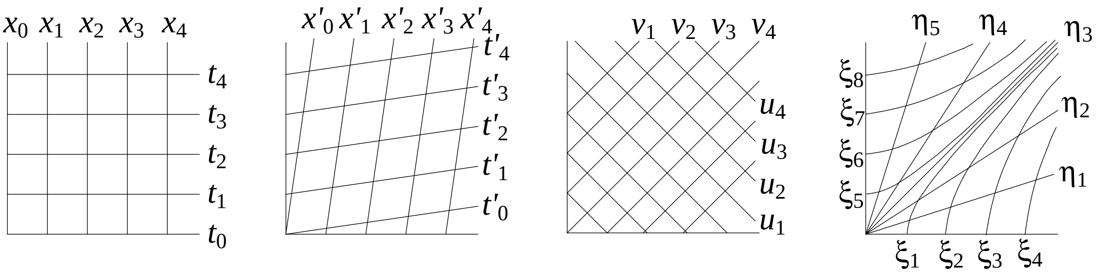

Fig. 1.— Four coordinate systems for the spacetime plane.

The coordinate transformation relating (t', x') to (t, x) is the Lorentz transformation of equation (1). The other two transformations are:

$$u = t + x \ , \ \ v = t - x \ , \tag{4}$$

and

$$\xi = \pm |t^2 - x^2|^{1/2} \ , \ \ \eta = \arctan(t/x) \ . \tag{5}$$

In this last case, we choose $\xi = +\sqrt{x^2 - t^2}$ if $x > t$ and $\xi = -\sqrt{t^2 - x^2}$ if $t > x$.

The last two coordinate systems in Figure 1 are unusual but there is nothing wrong with them. They assign unique labels to each event (point in the spacetime plane), and if two events have values of the coordinates that are very close, then the events are very close. The reverse, however, is not true everywhere for the fourth coordinate system: there exist events that are nearby in the plane yet have very different (η, ξ) coordinate values. Can you see where?

As an exercise in Assignment 2, you will work out the metric of Minkowski space for the coordinate systems of Figure 1. The first case is simply $d\tau^2 = dt^2 - dx^2$. The other cases follow by coordinate transformation. They have no particular physical significance; they are simply examples of non-Cartesian coordinate systems of bookkeeper coordinates.

4. Example: Coordinates and Distances on the Sphere

Next we consider the simplest case of a curved manifold, a two-dimensional sphere like the surface of the earth. Everyone is familiar with latitude and longitude on the earth; they are coordinates for the sphere. We call the longitude ϕ and the colatitude (i.e. the angle measured southward from the north pole) θ. If we have two points on a sphere of radius 1, with coordinates (θ_1, ϕ_1) and (θ_2, ϕ_2), what is the distance between them? You will answer this question in Assignment 2, without assuming that the two points are close to each other.

When the two points are very close, the distance between them gives the incremental distance on the unit sphere:

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \ . \tag{6}$$

It is conventional to call this formula *the metric* for a sphere. Thus, a metric exists for *any* manifold which has a rule for computing distances. (An example of a manifold *without* a metric is the P-V diagram of thermodynamics.)

In equation (6) we use the symbol ds for proper distance, i.e. the physical distance between two nearby points (as opposed to the coordinate differential $d\theta$ or $d\phi$). Notice that the Pythagorean theorem does not work on a sphere: the distance is *not* $(d\theta)^2+(d\phi)^2$. That is because the coordinates are not Cartesian. In fact, *there is no Cartesian coordinate system for the sphere!*

You might object that the Pythagorean theorem holds if we include the fact that the sphere exists in three dimensions, and we then use Cartesian coordinates to label the points. However, in this case, one is no longer considering the sphere by itself — one is considering a sphere embedded in three-dimensional Euclidean space. We don't want to do that. We are concerned about the *intrinsic* geometry of the two-dimensional surface of the sphere, and this requires only two coordinates. The point is, the intrinsic geometry is non-Euclidean.

Although it is impossible to find a two-dimensional globally Euclidean coordinate system for the sphere (i.e., coordinates for which the Pythagorean theorem holds everywhere), it is possible to find coordinates that are *locally* Euclidean. That is why maps of cities and even countries can be made on a flat page with little distortion. As an example, near the equator we can write $x = \phi$, $y = \frac{\pi}{2} - \theta$. The metric is

$$ds^2 = dy^2 + \cos^2 y\, dx^2 \approx dy^2 + dx^2 \ , \tag{7}$$

where the corrections are $O(y^2)$. (That is, as $y \to 0$, the correction terms in the metric go to zero at least as fast as y^2.) This means that, as long as y is not too large (i.e. one is not far from the equator), the coordinates are almost Cartesian. If one wishes nearly Cartesian coordinates that work far away from the equator, it is only necessary to rotate the sphere to define a new equator.

For $dy = 0$, the physical distance along a meridian of longitude is $ds = (\cos y)dx$. One must not think of this physical distance as a coordinate distance (i.e. a bookkeeper distance). Mathematically, one cannot define a coordinate by $d\xi = (\cos y)dx$ and integrate it to get Euclidean coordinates (ξ, y). (The $\cos y$ is the problem; y is not a function of x.) There is no coordinate system on the sphere for which the Pythagorean theorem holds everywhere. By extension of this argument, one can always find a *local inertial frame* in curved spacetime, but no inertial frame is global.

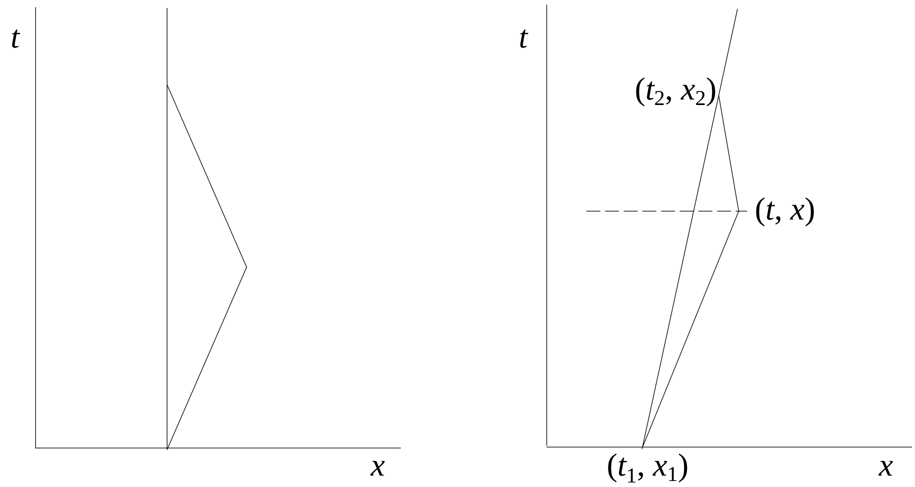

Fig. 2.— Two views of the twin paradox, showing the worldlines of a twin moving at constant speed and another who changes velocity during the trip. Left: one twin remains at rest while the other takes a roundtrip flight. Right: both twins move, although one has constant speed. The intermediate point may be moved anywhere along the dashed line; the text shows which path has the maximal proper time.

5. Maximize your Proper Time: The Twin Paradox

As a last example of coordinates and proper time, we examine the classic twin paradox of special relativity. The left panel of Figure 2 shows the trajectories of two twins in a Minkowski (spacetime) diagram. One twin stays home and ages by $t_2 - t_1 \equiv T$. Another twin travels at high speed, first away from the earth and then back, and ages by T/γ where $\gamma > 1$ is the Lorentz factor.

Note that the age is *not* the total length of the trajectory computed using the Pythagorean theorem, even in special relativity. That is because, as equation (2) shows, space and time terms have opposite signs in the metric. Both twins have the same elapsed bookkeeper time t. However, the stay-at-home twin has $dx = 0$ while the moving twin has $dx \neq 0$ and therefore has less elapsed proper time. The moving twin's proper time is the same as the time measured by a wristwatch or any clock that she carries. The stationary twin ages more. If you are unfamiliar with the paradox and its resolution, please review a special relativity book.

The upshot of this thought experiment is that in the absence of forces, a particle takes the path of greatest proper time between fixed endpoints in a Minkowski diagram. To see this, consider a different example, shown in the right panel of Figure 2. Now the first twin has a constant speed, i.e. a constant slope in the spacetime diagram. The second twin first travels to some spatial coordinate

x, then reverses course to meet the other twin. Which twin ages more?

To answer this question, we let x be the variable x-coordinate of the turn-around event, and we compute the age of the second twin as a function of x. We will see that this function has exactly one maximum, and it occurs when x is chosen (with t fixed) so that the two paths overlap.

The age of the second twin is

$$\tau(x) = \frac{t - t_1}{\gamma_1} + \frac{t_2 - t}{\gamma_2} \quad \text{where} \quad \gamma_1 = (1 - v_1^2)^{-1/2} \ , \ \gamma_2 = (1 - v_2^2)^{-1/2} \ . \tag{8}$$

Using $v_1 = (x - x_1)/(t - t_1)$ and $v_2 = (x_2 - x)/(t_2 - t)$, we see that γ_1 and γ_2 are functions of x, everything else being a constant. It follows that

$$\frac{d\tau}{dx} = \gamma_2 v_2 - \gamma_1 v_1 \ . \tag{9}$$

This vanishes if and only if $v_1 = v_2$, i.e. the second twin has constant speed so her worldline is the straight one shown in the figure. Taking the second derivative, one can show that $d^2\tau/dx^2 < 0$. Therefore, the case $v_1 = v_2$ is a maximum. *For fixed endpoints in a spacetime diagram, the path of constant speed is the path of maximal proper time.* In Chapter 2 of EBH, this is elevated to a principle: when no non-gravitational forces act on a body, the body follows a path of maximal proper time. For photons this can be a path of minimal proper time, so to cover all cases it is called the principle of extremal aging. (An extremum of a function is a place where its first derivatives are zero: a maximum, a minimum, or a saddle point.)