Lecture (5)

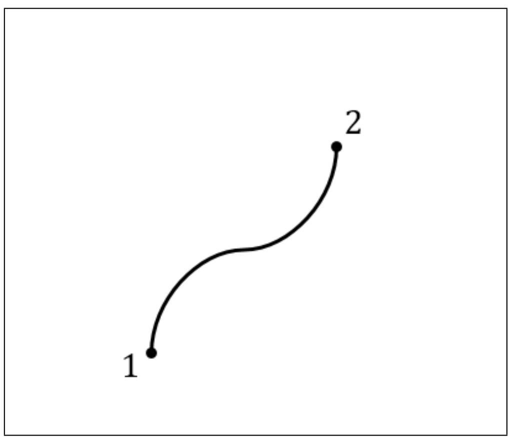
Today: Conserved Quantities and Center of Mass

For Next Lecture

1. read LL 9-11
2. do problems 11-14

1 Conservation of Energy

In 8.01 you encountered conservation of energy as an integral along the path of a particle, which gave the work done on it.



8.01 Energy Conservation

$$\begin{aligned} W_{12} &= \int_1^2 \vec{F}\cdot d\vec{r} = \int_{t_1}^{t_2} (m\frac{\mathrm{d}}{\mathrm{d}t}\vec{v})\cdot\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\,dt \\ &= m\int_{t_1}^{t_2}\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\cdot\vec{v}\,dt = \frac{1}{2}m\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}(\vec{v}\cdot\vec{v})\,dt \\ &= \frac{1}{2}m(v_2^2 - v_1^2) = \Delta T \end{aligned}$$

So the work performed lead to a change in kinetic energy. For a conservative force, we can also relate this to the change in potential energy...

$$
\begin{aligned}
W_{12} &= \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 (-\nabla U) \cdot d\vec{r} = U_1 - U_2 = -\Delta U \\
&\Rightarrow \Delta T = -\Delta U \Rightarrow \Delta(T+U) = 0
\end{aligned}
$$

$E = T + U$ is conserved!

In the context of Lagrangian Mechanics and the PLA, on the other hand, no vector calculus is required. We need only assume that *time is homogeneous*, that is that L has no explicit time dependence.

Lagrangian Energy Conservation

$$
\begin{aligned}
\text{given } \frac{\partial L}{\partial t} &= 0 \text{ i.e. time is homogeneous} \\
\frac{\mathrm{d}}{\mathrm{d}t} L(q, \dot{q}) &= \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \leftarrow \text{ total time derivative} \\
\Rightarrow \frac{\mathrm{d}L}{\mathrm{d}t} &= \dot{q}\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial \dot{q}}) + (\frac{\mathrm{d}}{\mathrm{d}t}\dot{q})\frac{\partial L}{\partial \dot{q}} \leftarrow \text{ used E-L} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\dot{q}\frac{\partial L}{\partial \dot{q}}) \leftarrow \text{ used product rule} \\
\Rightarrow 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}\frac{\mathrm{d}L}{\mathrm{d}\dot{q}} - L\right) \\
\Rightarrow E &\equiv \dot{q}\frac{\partial L}{\partial \dot{q}} - L = \text{constant in time}
\end{aligned}
$$

hmm... did we really do anything there? Profoundly simple or totally vacuous?

Note:

$$\begin{aligned} T &= \frac{1}{2}m\sum_{jk} a_{jk}\dot{q}_j\dot{q}_k \text{ and } L = T(q,\dot{q}) - U(q) \\ \dot{q}_i\frac{\partial L}{\partial \dot{q}_i} &= \frac{1}{2}m\dot{q}_i\sum_{jk} a_{jk}(\delta_{ij}\dot{q}_k + \delta_{ik}\dot{q}_i) \\ &= \frac{1}{2}m\sum_{jk} a_{jk}(\dot{q}_j\dot{q}_k + \dot{q}_k\dot{q}_j) = 2T \\ \Rightarrow E &= 2T - (T-U) = T+U \\ \text{given} & \quad \text{homogeneous time and a conservative potential} \end{aligned}$$

So, conservation of energy is built into the E-L equations. As long as there are no time varying forces (e.g. external drivers) or velocity dependent forces (e.g. friction) energy is conserved! (Actually velocity dependent potentials are ok, E is still conserved, but $E = T + U$ is not guaranteed! More on velocity dependent potentials later in the course.)

---

2 Conservation of Momentum

Conservation of momentum, in 8.01, was found by integrating in time (rather than space as we did for energy). With a single particle and no external forces, this is very simple.

Conservation of Momentum

$$\int_{t_1}^{t_2} \vec{F}\,dt = \int_{t_1}^{t_2} \dot{p}\,dt = \Delta p = 0 \text{ if } F = 0 \text{ (in 8.01)}$$

Single Particle Conservation of Momentum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \Rightarrow \frac{\partial L}{\partial \dot{q}} = \text{ const } \text{ if } \frac{\partial L}{\partial q} = 0 \text{ (in 8.223)}$$

we have already seen that we need only assume that space is homogeneous and isotropic to show that a free particle moves at constant velocity in Lagrangian Mechanics. What about a system of particles?

$$\text{no external forces} \;\Rightarrow\; \sum_n \vec{F}_n = 0$$
$$\Rightarrow \sum_n \nabla_n U(\vec{r}_1, ..., \vec{r}_N) = 0 \;\Rightarrow\; \sum_n \frac{\partial U}{\partial \vec{r}_n} = 0$$

$$\text{Note: } \nabla_n U(\vec{r_1}, ..., \vec{r_N}) = \frac{\partial U}{\partial x_n}\widehat{x} + \frac{\partial U}{\partial y_n}\widehat{y} + \frac{\partial U}{\partial z_n}\widehat{z}$$

That is 8.01 style. Let's do this 8.223 style by requiring homogeneous and isotropic space. (Homogeneity of a scalar field $\Rightarrow$ constant $\Rightarrow$ isotropic). If space is homogeneous, a small shift of all particle coordinates should not change the Lagrangian.

$$\vec{r}_n^{\,\prime} = \vec{r}_n + \vec{\epsilon} \qquad \text{where } \epsilon \text{ is a small constant offset}$$
$$L' = L(\vec{r}^{\,\prime}, \dot{\vec{r}}^{\,\prime}, t) = L(\vec{r}, \dot{\vec{r}}, t) + \sum_n \left(\frac{\partial L}{\partial \vec{r}_n} \cdot \vec{\epsilon} + \frac{\partial L}{\partial \dot{\vec{r}}_n} \cdot \dot{\vec{\epsilon}} \right)$$
$$\Rightarrow L' - L = \sum_n \frac{\partial L}{\partial \vec{r}_n} \cdot \vec{\epsilon} = \vec{\epsilon} \cdot \sum_n \frac{\partial L}{\partial \vec{r}_n}$$
$$L' - L = 0 \;\Rightarrow\; \sum_n \frac{\partial L}{\partial \vec{r}_n} = 0$$

That is, the sum of generalized forces over all particles is zero.

What about the total momentum of the particles? Summing E-L over particles:

$$\sum_n \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\vec{r}}_n}\right) = \sum_n \frac{\partial L}{\partial \dot{\vec{r}}_n} = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\sum_n \frac{\partial L}{\partial \dot{\vec{r}}_n} = 0$$
$$\Rightarrow \vec{P} \equiv \sum_n \frac{\partial L}{\partial \dot{\vec{r}}_n} = \sum_n \vec{p}_n = \text{ constant}$$

So the total generalized momentum is conserved in an isolated system (no external forces, homogeneous space).

Note: unlike E, P does not depend on the interaction potential. It is just the sum of each particle's $\vec{p}_n$.

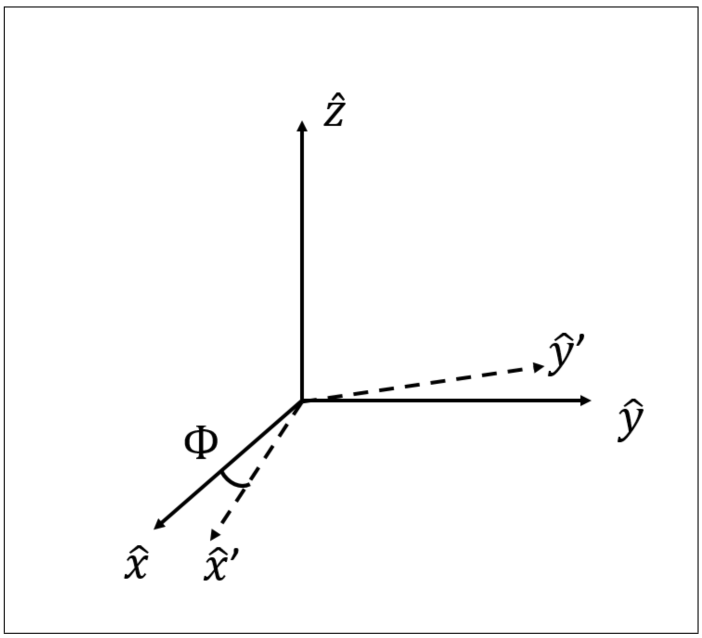
---

3 Conservation of Angular Momentum

We have seen that time invariance of the L leads to Energy Conservation, and translation invariance of L leads to Momentum Conservation. What about angular momentum?

Angular Momentum and Rotational Invariance of L

$$x' = x\cos\phi - y\sin\phi$$
$$y' = y\cos\phi + x\sin\phi$$
$$\text{for } \phi \ll 1, \sin\phi \approx \phi, \cos\phi \approx 1$$
$$x' \approx x - y\phi\ ,\ y' = y + x\phi$$
$$z' = z \Rightarrow \vec{\epsilon} = \{ - y\phi, x\phi, 0$$



$$
\begin{aligned}
L' = L(\vec{r}\,', \dot{\vec{r}}\,') &= L(\vec{r} + \vec{\epsilon}, \dot{\vec{r}} + \dot{\vec{\epsilon}}) \\
&= L(\vec{r}, \dot{\vec{r}}) - \frac{\partial L}{\partial x} y\phi + \frac{\partial L}{\partial y} x\phi - \frac{\partial L}{\partial \dot{x}} \dot{y}\phi + \frac{\partial L}{\partial \dot{y}} \dot{x}
\end{aligned}
$$

$$
\text{Rotational Invariance} \;\Rightarrow\; L' - L = 0
$$

$$
\begin{aligned}
\Rightarrow 0 &= \phi(\dot{p_y} x + p_y \dot{x} - (\dot{p_x} y + p_x \dot{y})) \\
0 &= \phi \frac{\mathrm{d}}{\mathrm{d}t}(p_y x - p_x y) \text{ for any small } \phi \\
\Rightarrow p_y x - p_x y &= \text{ constant } \Rightarrow L_z = \text{ constant} \\
\text{NB: } \vec{L} &= \vec{r} \times \vec{p} \Rightarrow L_z = x\, p_y - y\, p_x
\end{aligned}
$$

Invariance to rotation about z-axis leads to conservation of z-component of angular momentum!

Of course there is nothing special about $\widehat{z}$; invariance to rotation about any axis implies conservation of angular momentum about that axis. If L is invariant to all rotations about a point (e.g. the origin), then $\vec{L}$ is conserved.

if U($|\vec{r}|$) "central potential", then $\vec{L}$ about origin is conserved

Note that like E and $\vec{P}$, $\vec{L}$ is additive and frame dependent.

LL uses $\vec{M}$ for AM, presumably to avoid confusion with Lagrangian. We will use $\vec{L}$ as is customary.

4 Center of Mass

While both E and p are deeply meaningful quantities in physics, they both depend on your choice of reference frame. There is, however, a "special" frame for any given system: the Center of Mass frame which has $p = 0$ and minimal Energy.

Center of Mass Frame

$P_{CM} = 0$, $E_{CM} = E_{\text{internal}}$ = minimum energy

To investigate this, let's take 2 frames with relative velocity Δv

$$\begin{aligned} v'_n &= v_n + \Delta v \quad \text{velocity of the n}^{th} \text{ particle} \\ P' &= \sum_n m_n(v_n + \Delta v) = P + \Delta v \sum_n m_n \\ P' &= P + M\Delta v \quad \text{where } M = \sum_n m_n \end{aligned}$$

Note that LL uses μ for total mass and m for reduced mass. It is much more common to use M for total mass and μ for reduced mass. The velocity of the CM frame is

$$\vec{v}_{CM} = \frac{\vec{P}}{M} \quad \text{velocity of CoM}$$

$$\vec{R}_{CM} = \frac{\sum_n m_n \vec{r}_n}{M} \quad \text{position of CoM}$$

The potential energy of a system of interacting particles is unaltered by a change in reference frame (since only relative positions of the particles matter), but the kinetic energy changes with velocity.

$$\begin{aligned} \text{if } T &= \frac{1}{2}\sum_n m_n v_n^2 \Rightarrow T' = \frac{1}{2}\sum_n m_n(\vec{v_n} + \Delta\vec{v})^2 \\ T' &= \frac{1}{2}\sum m_n(v_n^2 + 2\vec{v_n}\cdot\Delta\vec{v} + \Delta v^2) \\ &= T + \frac{1}{2}M\Delta v^2 + \Delta\vec{v}\cdot\vec{P} \end{aligned}$$

If the initial frame was the CoM frame with $\vec{P} = 0$, then

$$\begin{aligned} \text{if } \vec{P} &= 0, \quad \text{(i.e. start in the CoM frame) then} \\ T' &= T + \frac{1}{2} M \Delta v^2 \end{aligned}$$

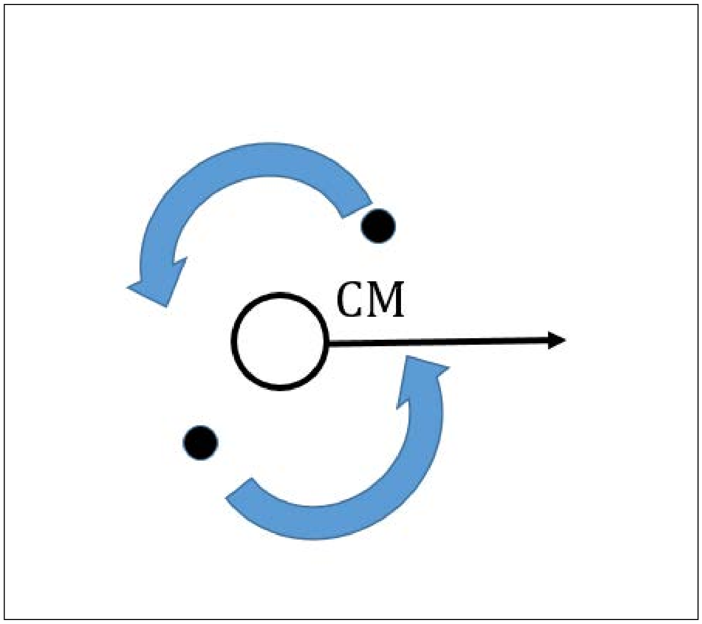
Such that the CoM frame is the one with minimum energy.

For a wide array of physics problems, the first step to a solution is a move to the CoM frame.

---

5 Frame Dependence of $\vec{L}$

$\vec{L} = \vec{L}_i + \vec{R_{cm}} \times \vec{P}$, where $\vec{L}_i$ is the AM in the CoM frame



So like energy total AM may be non-zero in the CoM frame. Unlike energy, there exists a frame with $\vec{L} = 0$, though I don't know how it is used or called...

Let's back up a moment and look at the big picture. For any isolated system of interacting particles we have 7 conserved quantities ($E + \vec{P} + \vec{L}$). By "isolated" I mean the Lagrangian doesn't depend on location, orientation or time. This is trivially true if the potential depends only on the relative positions of the particles (e.g. due to gravity or electric charge).

---

6 Homogeneous and Isotropic

Homogeneous and isotropic space

$U(\vec{r_1}, \vec{r_2}) = \alpha |\vec{r_1} - \vec{r_2}|^2$

Homogeneous, but NOT isotropic (vector field only)

$\vec{F}(\vec{r_1}, \vec{r_2}) = \alpha\,\widehat{x}$ uniform force in $\widehat{x}$ direction

Isotropic about origin (NOT homogeneous)

$U(\vec{r_1}, \vec{r_2}) = \alpha r_1^2 + \beta r_2^2$ central force, origin is special

For Next Lecture

1. *read LL 9-11*
2. *do problems 11-14*

8.223 Classical Mechanics II
January IAP 2017