

(18) Connections

Last time we looked at canonical transforms and generator functions.

Let's look at another interesting type of transform: the infinitesimal transform (aka, how to take small steps).

Infinitesimal Transform:

$$F_2(q, P) = qP + \varepsilon G(q, P)$$
$$p = \frac{\partial F_2}{\partial q} = P + \varepsilon \frac{\partial G}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P} = q + \varepsilon \frac{\partial G}{\partial P}$$

$$\lim_{\varepsilon \rightarrow 0} \frac{\partial G(q, P)}{\partial P} = \frac{\partial G(q, p)}{\partial p} \text{ since } P \rightarrow p$$
$$P = p - \varepsilon \frac{\partial G}{\partial q}, \quad Q = q + \varepsilon \frac{\partial G}{\partial p}$$

Let $\varepsilon = dt$, $G = H$

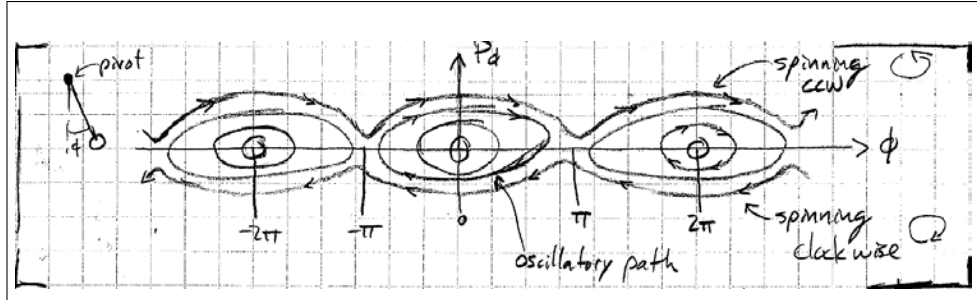
$$P = p - \frac{\partial H}{\partial q} dt = p + \dot{p} dt, \quad Q = q + \dot{q} dt$$
$$P(t) = p(t + dt), \quad Q(t) = q(t + dt)$$

This canonical transform actually takes us forward in time by a little step dt !

“ H is the generator of time translation”

Since using the Hamiltonian as a generator function moves the system in time. This is the basis of the SE in 8.04.

We can also make a connection now to 8.044. Statistical mechanics relies heavily on the idea of “phase space” which is derived from Hamiltonian mechanics. We can think of our generalized coordinates and momenta as defining a $2N$ dimensional space through which the state of the particle moves in time.



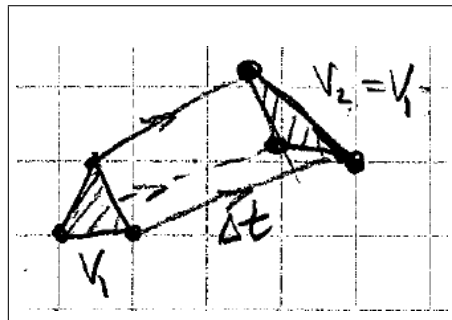
This picture is fundamental because volumes in phase space do not change in time. To show that volumes are constant, we use the Hamiltonian.

Liouville's Theorem:

$$\nabla \cdot \left\{ \dot{\vec{p}}, \dot{\vec{q}} \right\} = \sum_i \frac{\partial}{\partial p_i} \dot{p}_i + \frac{\partial}{\partial q_i} \dot{q}_i = 0$$

since $\frac{\partial}{\partial p} \dot{p} + \frac{\partial}{\partial q} \dot{q} = -\frac{\partial}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial}{\partial q} \frac{\partial H}{\partial p} = 0$

This is known as Liouville's Theorem. The result is that ensembles of particles maintain their phase space volume in time.



The concept of phase space was developed in the late 19th century by Boltzmann, Poincare, and Gibbs; the founders of classical statistical mechanics. In statistical mechanics, for a system in equilibrium it is assumed that every microstate consistent with the current macrostate has an equal probability. That is, all regions of phase space consistent with the current macrostate have equal probability density. If that is initially true, then Liouville's theorem tells us it will remain true. If that were not the case, if probability density increased in some regions and decreased in others, then it would be impossible to make this assumption. Even if it were true at one time, it would not be true a moment later.

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